

Efficient Inference and Computation of Optimal Alternatives for Preference Languages Based On Lexicographic Models

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Nic Wilson, Anne-Marie George

Insight Centre for Data Analytics, School of Computer Science and IT

University College Cork, Ireland

{nic.wilson, annemarie.george}@insight-centre.org

Abstract

We analyse preference inference, through consistency, for general preference languages based on lexicographic models. We identify a property, which we call *strong compositionality*, that applies for many natural kinds of preference statement, and that allows a greedy algorithm for determining consistency of a set of preference statements. We also consider different natural definitions of optimality, and their relations to each other, for general preference languages based on lexicographic models. Based on our framework, we show that testing consistency, and thus inference, is polynomial for a specific preference language $\mathcal{L}'_{pqT}$, which allows strict and non-strict statements, comparisons between outcomes and between partial tuples, both *ceteris paribus* and strong statements, and their combination. Computing different kinds of optimal sets is also shown to be polynomial; this is backed up by our experimental results.

1 Introduction

Preferences are considered in many different fields such as recommender systems and human-computer interaction [Chen and Pu, 2007; Trabelsi *et al.*, 2013], databases [Agrawal and Wimmers, 2000; Kießling, 2002] and multi-objective decision making and social choice [Arrow and Raynaud, 1986; Lang, 2002; Sandholm and Boutilier, 2006]. They are used to, e.g., find optimal decisions, solutions or items that satisfy the wishes or needs of a user or a group. In order to reason with preferences, typically assumptions are made on the form of the relation used to model the user preferences. In this paper we restrict our considerations to lexicographic models, and we consider preference inference, which involves reasoning about the whole set of preference models that are consistent with the input preferences, as in [Kohli and Jedidi, 2007; Wilson, 2014]; this contrasts with work that focuses on learning one lexicographic model that fits best with the given preferences [Booth *et al.*, 2010; Bräuning and Hüllermeier, 2012; Dombi *et al.*, 2007; Yaman *et al.*, 2010].

Preference inference aims to overcome gaps of knowledge in the user preferences by analysing the given preferences

and deciding whether another preference statement can be deduced. Since in this paper we allow negations of statements, preference inference can be reduced to testing consistency of a set of preference statements. We define a property, strong compositionality, that is satisfied by many natural types of preference statements and enables a simple greedy algorithm for testing consistency.

Fast computation is essential in many applications; we give a concrete instance $\mathcal{L}'_{pqT}$ of a preference language that allows polynomial computation. This can represent a relatively expressive form of preference input. It can express preferences between a pair of complete assignments to a set of variables, and, more generally, between partial assignments to just a subset of the variables, allowing *ceteris paribus* assumptions or stronger implications on the values of other variables. We also allow non-strict statements, two forms of strict preference statements and can represent certain negated statements.

Previous work on preference inference based on standard lexicographic models have considered more restricted preference languages. Wilson [2014] considered only non-negated non-strict statements, which can only express that one assignment is at least as good as another (or equivalent). Kohli and Jedidi [2007] considers only non-negated strict statements, which can only express that one complete assignment is strictly better than another. In [Wilson *et al.*, 2015a], preference statements are comparisons of complete assignments, and fixed value orders for the variables are given.

Other work on preference inference includes that based on weighted sum models [Wilson *et al.*, 2015b; Montazery and Wilson, 2016], on hierarchical models [Wilson *et al.*, 2015a], conditional lexicographic models [Wilson, 2009]; Pareto orders [George and Wilson, 2016], and on general strict total orders, as in e.g., work on conditional preference structures such as [Boutilier *et al.*, 2004]. While lexicographic models prevent tradeoffs between variables, they usually allow better complexity results for general preference languages.

Given a set of preference inputs and a set of alternatives, there are several natural notions of optimal set; for instance being undominated with respect to the induced preference relation; or being possibly optimal, i.e., optimal in at least one model. We establish relationships between the different notions of optimality. For preference language $\mathcal{L}'_{pqT}$, we analyse the running times and set relations theoretically and experimentally, illustrating the efficiency of the algorithms.

The paper is organised as follows: Section 2 gives basic definitions. The concepts of inference, consistency and strongly compositional statements are investigated in Section 3. Section 4 introduces languages $\mathcal{L}_{pqT}$ and $\mathcal{L}'_{pqT}$ and an algorithm to decide consistency in this context. In Section 5, different notions of optimality are compared for general preference languages. The efficiency of computing these for $\mathcal{L}'_{pqT}$ is analysed theoretically and experimentally in Section 6. The last section concludes.

2 Lex Models, Composition and Extension

In this section, we give some basic definitions; in particular, we define lexicographic models, along with a natural composition operation and extension relation, that are important in our approach. A lexicographic model has an associated ordering of (some of the) variables, along with value orderings for each such variable. This generates a (lexicographic) ordering relation, by first comparing outcomes α and β on the first (i.e., most important) variable Y ; only if $\alpha(Y) = \beta(Y)$ is the next most important variable considered.

Throughout the paper we consider a fixed set V of n variables, where for each $X \in V$, $D(X)$ is the set of possible values of X . For subset of variables $A \subseteq V$ let $\underline{A} = \prod_{X \in A} D(X)$ be the set of possible assignments to A . For $X \in V$, we abbreviate $\{X\}$ to $\underline{X}$ (which is essentially $D(X)$). An *outcome* is an element of $\underline{V}$, i.e., an assignment to all the variables. For outcome α and subset U of V , we define $\alpha(U)$ to be the tuple in $\underline{U}$ generated by projecting (i.e., restricting) α to U . These definitions describe a decision scenario over a set of alternatives given by different features. Consider for example the choice between different flight connections. A set of variables to describe the flight connections could be $V = \{\text{airline}, \text{time}, \text{class}\}$. The variable domains are then given by $D(\text{airline}) = \underline{\text{airline}} = \{KLM, LAN\}$, $D(\text{time}) = \{\text{day}, \text{night}\}$ and $D(\text{class}) = \{\text{economy}, \text{business}\}$. Then projection of the outcome $\alpha = (KLM, \text{day}, \text{economy})$ to the variables $U = \{\text{airline}, \text{class}\}$ is given by $\alpha(U) = (KLM, \text{economy})$.

Define $\mathcal{G}$ to be the set of lexicographic models (over the set of variables V); a *lexicographic model* or *lex model*, π (over V), is defined to be a (possibly empty) sequence of the form $(Y_1, \geq_{Y_1}), \dots, (Y_k, \geq_{Y_k})$, where Y_i ($i = 1, \dots, k$) are different variables in V , and each $\geq_{Y_i}$ is a total order on Y_i . The associated relation $\succsim_\pi \subseteq \underline{V} \times \underline{V}$ is defined as follows: for outcomes α and β , $\alpha \succsim_\pi \beta$ if and only if either (i) for all $i = 1, \dots, k$, $\alpha(Y_i) = \beta(Y_i)$; or (ii) there exists $i \in \{1, \dots, k\}$ such that for all $j < i$, $\alpha(Y_j) = \beta(Y_j)$ and $\alpha(Y_i) \succ_{Y_i} \beta(Y_i)$ (i.e., $\alpha(Y_i) \geq_{Y_i} \beta(Y_i)$ and $\alpha(Y_i) \neq \beta(Y_i)$). Thus $\succsim_\pi$ is a total pre-order on $\underline{V}$, which is a total order if $k = n = |V|$.

The corresponding strict relation $\succ_\pi$ is given by $\alpha \succ_\pi \beta$ if and only if there exists $i \in \{1, \dots, k\}$ such that $\alpha(Y_i) \succ_{Y_i} \beta(Y_i)$ and for all $j < i$, $\alpha(Y_j) = \beta(Y_j)$. The corresponding equivalence relation $\equiv_\pi$ is given by $\alpha \equiv_\pi \beta$ if and only if for all $i = 1, \dots, k$, $\alpha(Y_i) = \beta(Y_i)$. Thus, $\alpha \equiv_\pi \beta$ if and only if $\alpha(V_\pi) = \beta(V_\pi)$, where $V_\pi = \{Y_1, \dots, Y_k\}$ is set of the variables involved in π . Consider lexicographic models $\pi = (\text{airline}, KLM > LAN), (\text{time}, \text{day} > \text{night})$ and $\pi' = (\text{class}, \text{economy} > \text{business}), (\text{time}, \text{night} > \text{day})$ for the

previous example of flight connections. In π the choice of airline decides which connection is preferred; only if these are the same, is the flight time considered. π' first compares the classes; only if these are the same, are the flight times considered. For outcomes $\alpha = (KLM, \text{day}, \text{economy})$, $\beta = (KLM, \text{night}, \text{business})$ and $\gamma = (LAN, \text{day}, \text{economy})$ we have $\alpha \succ_\pi \beta \succ_\pi \gamma$ and $\alpha \equiv_{\pi'} \gamma \succ_{\pi'} \beta$.

Composition of lexicographic models: We define an important composition operation on lex models, which can be shown to be associative. Let $\pi = (Y_1, \geq_{Y_1}), \dots, (Y_k, \geq_{Y_k})$, and $\pi' = (Z_1, \geq_{Z_1}), \dots, (Z_l, \geq_{Z_l})$ be two lexicographic models. Let π'' be the sequence π' but where pairs $(Z_i, \geq_{Z_i})$ are omitted if $Z_i \in V_\pi$. Define lex model $\pi \circ \pi'$ to be π followed by π'' . Note that $V_{\pi \circ \pi'} = V_\pi \cup V_{\pi'}$.

Lemma 1. $\circ$ is an associative operation on $\mathcal{G}$.

Proof. For $\pi, \pi' \in \mathcal{G}$, we say that π and π' are disjoint if $V_\pi \cap V_{\pi'} = \emptyset$. For set of variables $U \subseteq V$, define $\pi^{\downarrow-U}$ to be π with all pairs of the form $(X, \geq_X)$ omitted, when $X \in U \cap V_\pi$. For any $\pi, \pi' \in \mathcal{G}$, we have that $\pi \circ \pi' = \pi \circ (\pi')^{\downarrow-V_\pi}$, by definition.

Consider any $\pi_1, \pi_2, \pi_3 \in \mathcal{G}$. For $j = 1, 2, 3$, let V_j equal V_{π_j} . Then, $(\pi_1 \circ \pi_2) \circ \pi_3 = (\pi_1 \circ \pi_2') \circ \pi_3'$, where $\pi_2' = (\pi_2)^{\downarrow-V_1}$ and $\pi_3' = (\pi_3)^{\downarrow-V_1 \cup V_2}$.

Associativity clearly holds for pairwise disjoint lexicographic models, so $(\pi_1 \circ \pi_2') \circ \pi_3' = \pi_1 \circ (\pi_2' \circ \pi_3')$. Now, $\pi_1 \circ (\pi_2 \circ \pi_3)$ equals $\pi_1 \circ (\pi_2 \circ \pi_3)^{\downarrow-V_1}$, so, to prove associativity of $\circ$ it is sufficient to prove that $(\pi_2 \circ \pi_3)^{\downarrow-V_1}$ equals $\pi_2' \circ \pi_3'$. We have that $(\pi_2 \circ \pi_3)^{\downarrow-V_1}$ equals $(\pi_2 \circ (\pi_3)^{\downarrow-V_2})^{\downarrow-V_1}$, which equals $\pi_2^{\downarrow-V_1} \circ \pi_3^{\downarrow-V_1 \cup V_2}$, i.e., $\pi_2' \circ \pi_3'$, completing the proof. $\square$

Definition 1. For lex models π and π' in $\mathcal{G}$ we say that π' extends π if $\pi' \neq \pi$ and the sequence π' begins with π . We then write $\pi' \sqsupset \pi$, and write $\pi' \sqsupseteq \pi$ to mean that π' extends or equals π . This holds if and only if there exists a lex model π'' such that $\pi' = \pi \circ \pi''$.

Lemma 2. For $\pi, \pi' \in \mathcal{G}$, we have $\pi' \sqsupseteq \pi$ if and only if $\pi \circ \pi' = \pi'$.

Proof. Suppose that π' extends or equals π , and write $\pi' = \pi \circ \pi''$ where $V_{\pi''} \cap V_\pi = \emptyset$. Then, $\pi \circ \pi' = \pi \circ (\pi \circ \pi'')$ which equals $\pi \circ \pi'' = \pi'$. Conversely, suppose that $\pi \circ \pi' = \pi'$. Let $k = |V_\pi|$ and let π'' be the lexicographic model consisting of the first k pairs of π' . This equals the first k pairs of $\pi \circ \pi'$, i.e., π , and thus, π' extends or equals π . $\square$

The composition of the two previously defined lex models is $\pi \circ \pi' = (\text{airline}, KLM > LAN), (\text{time}, \text{day} > \text{night}), (\text{class}, \text{economy} > \text{business})$ and $\pi \circ \pi' \sqsupseteq \pi$.

We have the following basic monotonicity property.

Lemma 3. For $\pi, \pi' \in \mathcal{G}$, suppose that π' extends π . If $\alpha \succ_{\pi'} \beta$ then $\alpha \succ_\pi \beta$. Also, if $\alpha \succ_\pi \beta$ then $\alpha \succ_{\pi'} \beta$.

Proof. If $\alpha \succ_\pi \beta$ then α and β differ on some variable in π , and α is better than β on the first such variable; thus the same holds for π' , since π' extends π , and so $\alpha \succ_{\pi'} \beta$ holds. Similarly, if $\beta \succ_{\pi'} \alpha$ then $\beta \succ_\pi \alpha$, which, using the fact that

$\succ_\pi$ and $\succ_{\pi'}$ are both weak orders, is equivalent to: if $\alpha \not\succ_\pi \beta$ then $\alpha \not\succ_{\pi'} \beta$, which thus implies: if $\alpha \succ_{\pi'} \beta$ then $\alpha \succ_\pi \beta$. $\square$

The result below is a key property of composition, which will be important later.

Lemma 4. *Let $\pi, \pi' \in \mathcal{G}$. If $\alpha \succ_\pi \beta$ and $\alpha \succ_{\pi'} \beta$ then $\alpha \succ_{\pi \circ \pi'} \beta$. If $\alpha \succ_\pi \beta$ and $\alpha \succ_{\pi'} \beta$ then $\alpha \succ_{\pi \circ \pi'} \beta$ and $\alpha \succ_{\pi' \circ \pi} \beta$. Suppose now that $\alpha \equiv_\pi \beta$. Then $\alpha \succ_{\pi'} \beta \iff \alpha \succ_{\pi \circ \pi'} \beta$; and $\alpha \succ_{\pi'} \beta \iff \alpha \succ_{\pi \circ \pi'} \beta$.*

Proof. Since $\pi' \circ \pi$ extends π' , we have, by Lemma 3, that (i) $\alpha \succ_{\pi'} \beta$ implies $\alpha \succ_{\pi' \circ \pi} \beta$. Similarly, (ii) if $\alpha \succ_\pi \beta$ then $\alpha \succ_{\pi \circ \pi'} \beta$, and thus also $\alpha \succ_{\pi \circ \pi'} \beta$. We also have (iii) if $\alpha \equiv_\pi \beta$ and $\alpha \succ_{\pi'} \beta$ then $\alpha \succ_{\pi \circ \pi'} \beta$. This is because α and β differ on some variable in $\pi \circ \pi'$, and the first such variable pair $(X, \geq)$ appears in π' ; and X is the earliest variable in π' that α and β differ on, so $\alpha \succ_{\pi'} \beta$ implies that $\alpha(X) > \beta(X)$ and thus, $\alpha \succ_{\pi \circ \pi'} \beta$.

Suppose that $\alpha \succ_\pi \beta$ and $\alpha \succ_{\pi'} \beta$. Then, by (i), $\alpha \succ_{\pi' \circ \pi} \beta$. If $\alpha \succ_\pi \beta$ then $\alpha \succ_{\pi \circ \pi'} \beta$, by (ii). So, we only need to deal with the case where $\alpha \equiv_\pi \beta$. Then $\alpha \succ_{\pi \circ \pi'} \beta$ follows by (iii).

Suppose that $\alpha \succ_\pi \beta$ and $\alpha \succ_{\pi'} \beta$. If $\alpha \succ_\pi \beta$ or $\alpha \succ_{\pi'} \beta$ then, by the above arguments, we have $\alpha \succ_{\pi \circ \pi'} \beta$, and thus also $\alpha \succ_{\pi \circ \pi'} \beta$. So, it just remains to deal with the case where $\alpha \equiv_\pi \beta$ and $\alpha \equiv_{\pi'} \beta$. But then $\alpha(V_\pi) = \beta(V_\pi)$ and $\alpha(V_{\pi'}) = \beta(V_{\pi'})$ and so, because $V_{\pi \circ \pi'} = V_\pi \cup V_{\pi'}$, we have $\alpha(V_{\pi \circ \pi'}) = \beta(V_{\pi \circ \pi'})$, proving $\alpha \equiv_{\pi \circ \pi'} \beta$ and hence $\alpha \succ_{\pi \circ \pi'} \beta$.

Assume now that $\alpha \equiv_\pi \beta$, so that $\alpha \succ_\pi \beta$ and $\beta \succ_\pi \alpha$. By the earlier parts we have $\alpha \succ_{\pi'} \beta \Rightarrow \alpha \succ_{\pi \circ \pi'} \beta$; and $\alpha \succ_{\pi'} \beta \Rightarrow \alpha \succ_{\pi \circ \pi'} \beta$. Since $\beta \succ_\pi \alpha$, we also have if $\alpha \not\succ_{\pi'} \beta$, i.e., $\beta \succ_{\pi'} \alpha$, then $\beta \succ_{\pi \circ \pi'} \alpha$, i.e., $\alpha \not\succ_{\pi \circ \pi'} \beta$. Similarly we have $\alpha \not\succ_{\pi'} \beta$ implying $\alpha \not\succ_{\pi \circ \pi'} \beta$. $\square$

3 Lexicographic Inference and Strongly Compositional Preference Statements

In this section we define *lexicographic inference*, which can also be expressed in terms of (lexicographic) consistency. We define a notion of *strong compositionality*, and show that this property enables a greedy algorithm for checking consistency, and hence for preference inference. More specifically, a preference statement φ is strongly compositional if the composition of two lex models satisfies φ whenever the second and some extension of the first satisfy φ .

Throughout this section, we assume some language $\mathcal{L}$, and satisfaction relation $\models \subseteq \mathcal{G} \times \mathcal{L}$. Here, the language $\mathcal{L}$ is a set of preference statements and the relation $\models$ describes when a lex model satisfies a statement. For example, we can consider the simple language $\mathcal{L}^O = \{\alpha \geq \beta \mid \alpha, \beta \in \underline{V}\}$ with the satisfaction relation $\pi \models \alpha \geq \beta \iff \alpha \succ_\pi \beta$ as described in the previous section. We extend the satisfaction relation to subsets Γ of $\mathcal{L}$ in the usual way: for $\Gamma \subseteq \mathcal{L}$, define $\pi \models \Gamma \iff \pi \models \varphi$ for each $\varphi \in \Gamma$. If $\pi \models \Gamma$ then we say that π *satisfies* Γ , or π is a [lexicographic] *model* of Γ (and similarly, for a single preference statement φ). We say that Γ is *consistent*, if there exists some $\pi \in \mathcal{G}$ satisfying Γ ; otherwise,

Γ is *inconsistent*. We will sometimes use negations of preference statements: when necessary we can always extend the language $\mathcal{L}$ and the relation $\models$, by defining $\pi \models \neg\varphi \iff \pi \not\models \varphi$, when $\varphi \in \mathcal{L}$.

A key problem is to determine if a set $\Gamma \subseteq \mathcal{L}$ of preference statements is consistent. We will derive methods for determining this in Section 3.3.

Some Basic Preference Statements

For outcomes $\alpha, \beta \in \underline{V}$, we can define preference statements $\alpha \geq \beta$ and $\alpha > \beta$, where we define $\pi \models \alpha \geq \beta \iff \alpha \succ_\pi \beta$, and $\pi \models \alpha > \beta \iff \alpha \succ_\pi \beta$. We also define preference statement $\alpha \equiv \beta$, with $\pi \models \alpha \equiv \beta \iff \alpha \equiv_\pi \beta$, which is if and only if $\pi \models \alpha \geq \beta$ and $\pi \models \beta \geq \alpha$.

We have that $\pi \models \alpha \geq \beta \iff \pi \not\models \beta > \alpha$, i.e., $\pi \models \neg(\beta > \alpha)$. So $\alpha \geq \beta$ and $\neg(\beta > \alpha)$ are equivalent preference statements, in that they are satisfied by exactly the same set of lex models. Similarly, $\alpha > \beta$ and $\neg(\beta \geq \alpha)$ are equivalent preference statements.

Consider the example of lex models π, π' and outcomes α, β, γ as before. Then $\pi \models \alpha \geq \beta$ and thus $\pi \not\models \beta > \alpha$. Also, $\pi' \models \alpha \equiv \gamma$, i.e., $\pi' \models \alpha \geq \gamma$ and $\pi' \models \gamma \geq \alpha$.

3.1 Lexicographic Inference

We define the lexicographic inference relation $\Gamma \models \varphi \iff \pi \models \varphi$ for all $\pi \in \mathcal{G}$ such that $\pi \models \Gamma$. For instance, we have, for $\Gamma \subseteq \mathcal{L}$ and $\alpha, \beta \in \underline{V}$, $\Gamma \models \alpha \geq \beta \iff \alpha \succ_\pi \beta$ holds for all $\pi \in \mathcal{G}$ such that $\pi \models \Gamma$. Similarly, $\Gamma \models \alpha > \beta \iff \alpha \succ_\pi \beta$ holds for all $\pi \in \mathcal{G}$ such that $\pi \models \Gamma$.

Lexicographic inference can be reduced to checking consistency, since we have $\Gamma \models \varphi \iff \Gamma \cup \{\neg\varphi\}$ is inconsistent; and $\Gamma \models \neg\varphi \iff \Gamma \cup \{\varphi\}$ is inconsistent. Thus, $\Gamma \models \alpha \geq \beta \iff \Gamma \cup \{\beta > \alpha\}$ is inconsistent; and $\Gamma \models \alpha > \beta \iff \Gamma \cup \{\beta \geq \alpha\}$ is inconsistent.

Consider flight connections as before. Let Γ be the set of statements $\{\alpha > \beta, \beta \geq \gamma\}$ with α, β, γ defined as in the previous section. Then the lex model π satisfies Γ , $\pi \models \Gamma$, and thus Γ is consistent. In fact, the only preference models that satisfy Γ are: $\pi = (\text{airline}, KLM > LAN), (\text{time}, \text{day} > \text{night})$, $\pi' = (\text{airline}, KLM > LAN), (\text{class}, \text{economy} > \text{business})$, and any extension of either of these two models. Thus for $\delta = (LAN, \text{night}, \text{business})$, $\Gamma \models \gamma > \delta$.

The Induced Relation $\models^*$ and Maximal Models

From the relation $\models$ we also define the derived relation $\models^*$ as follows, where $\pi \in \mathcal{G}$ and $\varphi \in \mathcal{L}$: $\pi \models^* \varphi$ if and only if there exists $\pi' \in \mathcal{G}$ either extending or equalling π such that $\pi' \models \varphi$. Thus, $\pi \models^* \varphi$ holds either if π satisfies φ or some extension of π satisfies φ . We extend the relation to sets of statements in the usual way: for $\Gamma \subseteq \mathcal{L}$, define $\pi \models^* \Gamma$ if and only if $\pi \models^* \varphi$ holds for every $\varphi \in \Gamma$. The following lemma follows easily from the definitions.

Lemma 5. *Let $\pi, \pi' \in \mathcal{G}$ and $\Gamma \subseteq \mathcal{L}$.*

(i) $\pi \models \Gamma \Rightarrow \pi \models^* \Gamma$.

(ii) *Suppose that π' extends π . Then $\pi' \models^* \Gamma \Rightarrow \pi \models^* \Gamma$.*

Proof. (i) follows immediately from the definition of $\models^*$. Regarding (ii), $\pi' \models^* \varphi$ implies that there exists π'' with $\pi'' \supseteq \pi'$ and $\pi'' \models \varphi$. Then, $\pi'' \supseteq \pi$, and thus, $\pi \models^* \varphi$. The

result then follows. $\square$

Definition 2. We say that $\pi \in \mathcal{G}$ is a maximal model of Γ if $\pi \models \Gamma$ and for all lex models π' extending π we have $\pi' \not\models \Gamma$.

Clearly, for $\pi \in \mathcal{G}$ such that $\pi \models \Gamma$, either π is a maximal model of Γ or there exists a maximal model of Γ that extends π . Thus, if Γ is consistent there exists a maximal model of Γ . Analogously, we define a maximal $\models^*$ -model of Γ to be an element $\pi \in \mathcal{G}$ such that (i) $\pi \models^* \Gamma$ and (ii) there does not exist π' extending π with $\pi' \models^* \Gamma$.

Consider preference statements Γ relating to flight connections α, β, γ as before. The lex model $(\text{airline}, KLM > LAN)$ satisfies $(\text{airline}, KLM > LAN) \models^* \Gamma$. The model $(\text{airline}, KLM > LAN), (\text{time}, \text{day} > \text{night}), (\text{class}, \text{business} > \text{economy})$ is a maximal model of Γ (and a maximal $\models^*$ -model of Γ).

3.2 Compositional and Strongly Compositional Preference Statements

We formulate a pair of properties of preference statements that have strong implications regarding lexicographic inference and optimality.

Definition 3 ((Strongly) Compositional). Let $\varphi \in \mathcal{L}$. We say that φ is compositional if for all $\pi, \pi' \in \mathcal{G}$,

$$\pi \models \varphi \text{ and } \pi' \models \varphi \text{ implies } \pi \circ \pi' \models \varphi.$$

We say that φ is strongly compositional if for all $\pi, \pi' \in \mathcal{G}$,

$$\pi \models^* \varphi \text{ and } \pi' \models \varphi \text{ implies } \pi \circ \pi' \models \varphi.$$

For $\Gamma \subseteq \mathcal{L}$, we define Γ to be compositional if every element of Γ is compositional. Similarly, we say that Γ is strongly compositional if every element of Γ is strongly compositional.

Note that if $\varphi \in \mathcal{L}$ is inconsistent (i.e., there exists no $\pi \in \mathcal{G}$ with $\pi \models \varphi$) then φ is trivially (strongly) compositional.

Instances of Strongly Compositional Statements

Although being strongly compositional might appear to be quite a restrictive assumption, it turns out that it is satisfied by many natural preference statements, as illustrated by the next proposition. First, we give a lemma which, roughly speaking, states that the property of being [strongly] compositional is closed under conjunction.

Lemma 6. Let $\Gamma \subseteq \mathcal{L}$ and let $\psi \in \mathcal{L}$. Suppose that ψ is such that for all $\pi \in \mathcal{G}$, $\pi \models \psi \iff \pi \models \Gamma$. If Γ is compositional then ψ is compositional. If Γ is strongly compositional then ψ is strongly compositional and, for all $\pi \in \mathcal{G}$, $[\pi \models^* \psi \iff \pi \models^* \Gamma \text{ and } \Gamma \text{ is consistent}]$.

Proof. Suppose that Γ is compositional. Consider any $\pi, \pi' \in \mathcal{G}$ with $\pi, \pi' \models \psi$. Then, $\pi, \pi' \models \Gamma$, which, since Γ is compositional, implies $\pi \circ \pi' \models \Gamma$, and thus, $\pi \circ \pi' \models \psi$, showing that ψ is compositional.

Now assume, for the remainder of the proof of this result, that Γ is strongly compositional. First suppose also that $\pi \models^* \psi$. Then there exists π'' such that $\pi'' \supseteq \pi$ and $\pi'' \not\models \psi$, which implies that ψ is consistent, and so Γ is consistent. Also, for all $\varphi \in \Gamma$ we have $\pi'' \models \varphi$ and hence, $\pi \models^* \varphi$. We have shown that $\pi \models^* \Gamma$.

For the converse, we assume that $\pi \models^* \Gamma$ and Γ is consistent. Consistency of Γ implies that there exists $\pi' \in \mathcal{G}$ such that $\pi' \models \Gamma$. Since Γ is strongly compositional, $\pi \circ \pi' \models \Gamma$, and so, $\pi \circ \pi' \models \psi$ which implies that $\pi \models^* \psi$, because $\pi \circ \pi' \supseteq \pi$.

We will finally show that ψ is strongly compositional. Consider any $\pi, \pi' \in \mathcal{G}$ with $\pi \models^* \psi$ and $\pi' \models \psi$. We have, by the earlier part, that $\pi \models^* \Gamma$, and also $\pi' \models \Gamma$. Γ being strongly compositional implies $\pi \circ \pi' \models \Gamma$ and thus, $\pi \circ \pi' \models \psi$, proving that ψ is strongly compositional. $\square$

Proposition 1. For any outcomes, $\alpha, \beta \in \underline{V}$, statements $\alpha \geq \beta$ and $\alpha > \beta$ are strongly compositional, and, for $\pi \in \mathcal{G}$, $\pi \models^* \alpha \geq \beta \iff \pi \models \alpha \geq \beta$. Also, if $\alpha \neq \beta$, $\pi \models^* \alpha > \beta \iff \pi \models \alpha > \beta$.

In addition, given $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, let $\varphi^{\mathfrak{R}}$ be some statement satisfying: $\pi \models \varphi^{\mathfrak{R}}$ if and only if $\succ_{\pi} \supseteq \mathfrak{R}$. Then $\varphi^{\mathfrak{R}}$ is strongly compositional, and, for any $\pi \in \mathcal{G}$, we have $\pi \models^* \varphi^{\mathfrak{R}} \iff \pi \models \varphi^{\mathfrak{R}}$.

Here, the second half of Proposition 1 follows from the first using Lemma 6. Further important examples of strongly compositional statements will be discussed in Section 4.

A Characterisation of Consistency for Strongly Compositional Γ

The definitions immediately imply the following.

Lemma 7. Let $\Gamma \subseteq \mathcal{L}$, and let $\pi, \pi' \in \mathcal{G}$.

- If Γ is strongly compositional then it is compositional.
- If Γ is compositional then $\pi \models \Gamma$ and $\pi' \models \Gamma$ imply $\pi \circ \pi' \models \Gamma$.
- If Γ is strongly compositional then $\pi \models^* \Gamma$ and $\pi' \models \Gamma$ imply $\pi \circ \pi' \models \Gamma$.

The last point implies that, for strongly compositional and consistent Γ , if $\pi \models^* \Gamma$ then there exists a model of Γ either equalling or extending π . In fact we have:

Lemma 8. Suppose that Γ is strongly compositional, and let π be an element of $\mathcal{G}$. Then [there exists $\pi' \in \mathcal{G}$ with $\pi' \supseteq \pi$ and $\pi' \models \Gamma$] if and only if [Γ is consistent and $\pi \models^* \Gamma$].

Proof. $\Rightarrow$: First assume that there exists $\pi' \in \mathcal{G}$ with $\pi' \supseteq \pi$ and $\pi' \models \Gamma$. Clearly, Γ is consistent. Consider any $\varphi \in \Gamma$. We have $\pi' \models \varphi$, which implies $\pi \models^* \varphi$. Therefore, $\pi \models^* \Gamma$.

$\Leftarrow$: Assume that Γ is consistent and $\pi \models^* \Gamma$. Then there exists $\pi' \in \mathcal{G}$ with $\pi' \models \Gamma$. Since Γ is strongly compositional, $\pi \circ \pi' \models \Gamma$, by Lemma 7, and we have $\pi \circ \pi' \supseteq \pi$. $\square$

Suppose that π' satisfies strongly compositional Γ and that π is a maximal $\models^*$ -model of Γ . Since we have $\pi \circ \pi' \models^* \Gamma$ (because we have $\pi \circ \pi' \models \Gamma$) and $\pi \circ \pi' \supseteq \pi$, then the maximality of π implies that $\pi = \pi \circ \pi'$, and thus, $\pi \models \Gamma$. We also have that $V_{\pi} \supseteq V_{\pi'}$. This implies the following theorem and corollary:

Theorem 1. If Γ is consistent and strongly compositional then every maximal $\models^*$ -model of Γ satisfies Γ . Also, if π and π' are two maximal $\models^*$ -models of Γ then $V_{\pi} = V_{\pi'}$.

Corollary 1. Let π be any maximal $\models^*$ -model of strongly compositional Γ . Then Γ is consistent if and only if $\pi \models \Gamma$.

3.3 Checking Consistency of a Set of Strongly Compositional Preferences

Corollary 1 shows that we can test consistency of strongly compositional Γ , by finding any maximal $\models^*$ -model π of it, and checking if π satisfies Γ . In this section we show how a maximal $\models^*$ -model of Γ can be constructed iteratively, starting from the empty model $\{\}$. We first check if $\{\} \models^* \Gamma$. The following lemma shows that this is equivalent to Γ being $\models^*$ -consistent, which is a very weak property, i.e., it just requires that each element of Γ is (individually) consistent.

Lemma 9. *Let $\Gamma \subseteq \mathcal{L}$. Γ is $\models^*$ -consistent (i.e., there exists some $\pi \in \mathcal{G}$ with $\pi \models^* \Gamma$) if and only if, for each $\varphi \in \Gamma$, φ is consistent. This also holds if and only if $\{\} \models^* \Gamma$.*

Proof. $\Rightarrow$: Suppose that Γ is $\models^*$ -consistent, and that $\pi \models^* \Gamma$. Then, for any $\varphi \in \Gamma$ there exists some $\pi' \in \mathcal{G}$ with $\pi' \supseteq \pi$ and $\pi' \models \varphi$, which implies that φ is consistent.

$\Leftarrow$: Consider any $\varphi \in \Gamma$. Since φ is consistent, there exists some $\pi \in \mathcal{G}$ with $\pi \models \varphi$, which entails that $\{\} \models^* \varphi$, where $\{\}$ is the empty lexicographic order. Then $\{\} \models^* \Gamma$, so Γ is $\models^*$ -consistent. $\square$

We say that π' *minimally extends* π if π' extends π and $|V_{\pi'}| = |V_{\pi}| + 1$, i.e., π' involves one more variable than π . The next lemma implies that in building a maximal $\models^*$ -model, we only need consider adding one variable at a time.

Lemma 10. *π is a maximal $\models^*$ -model of Γ if and only if $\pi \models^* \Gamma$ and there exists no π' minimally extending π such that $\pi' \models^* \Gamma$.*

Proof. $\Rightarrow$ is obvious. For the converse, assume that there exists no π' minimally extending π such that $\pi' \models^* \Gamma$; consider any π'' extending π . Then there exists π' that minimally extends π and such that π'' extends or equals π' . By the hypothesis, $\pi' \not\models^* \Gamma$, which implies, using Lemma 5(ii), that $\pi'' \not\models^* \Gamma$, proving that π is a maximal $\models^*$ -model of Γ . $\square$

Starting with the empty model we grow a maximal $\models^*$ -model of Γ , by (iteratively) replacing the model with one minimally extending it and still $\models^*$ -satisfying Γ , if such a model exists. Otherwise, we have a maximal $\models^*$ -model π of Γ . By Corollary 1, we can test if Γ is consistent by checking $\pi \models \Gamma$.

We say that a *satisfaction test* is a test of the form $\pi \models \varphi$ for some $\pi \in \mathcal{G}$ and $\varphi \in \mathcal{L}$; a $\models^*$ -*satisfaction test* is a test of the form $\pi \models^* \varphi$.

Suppose that d is an upper bound on the domain size of each variable, i.e., for each $X \in V$, $|\underline{X}| \leq d$. We first have to check that the empty model is a $\models^*$ -model of Γ ; this involves a number $|\Gamma|$ of $\models^*$ -satisfaction tests. At each iterative step, there are less than $nd!$ possible extensions π of the current model π' to an extra variable, and there are at most n iterative steps, so there are less than $|\Gamma|n^2d!$ $\models^*$ -satisfaction tests required in the whole iterative process. Finally we have to check that the produced maximal $\models^*$ -model π of Γ satisfies Γ , which involves $|\Gamma|$ satisfaction tests. In total we have at most $|\Gamma|$ satisfaction tests and $|\Gamma|n^2d!$ $\models^*$ -satisfaction tests.

For bounded domain size, testing consistency of strongly compositional Γ can be performed with polynomial number

of tests of the form $\pi \models \varphi$ or $\pi \models^* \varphi$. For this algorithm to be efficient we need that each preference statement allows efficient tests of $\pi \models \varphi$ and $\pi \models^* \varphi$. In Section 4 we define languages that satisfy these properties.

3.4 Equivalence w.r.t. Compositional Statements

The following result implies that if Γ is compositional then two maximal models of Γ involve exactly the same set of variables, and if two outcomes are equivalent in one maximal model of Γ then they are equivalent in all models of Γ .

Proposition 2. *Assume that $\Gamma \subseteq \mathcal{L}$ is consistent and compositional. Let π, π' be models of Γ with π a maximal model of Γ . Then, $V_{\pi} \supseteq V_{\pi'}$; and $V_{\pi} = V_{\pi'}$ if and only if π' is a maximal model of Γ . Also, if, for $\alpha, \beta \in \underline{V}$, we have $\alpha \equiv_{\pi} \beta$ then we have $\alpha \equiv_{\pi'} \beta$, and in fact $\Gamma \models \alpha \equiv \beta$.*

Proof. Let π be a maximal model of Γ and let π' be any model of Γ . Compositionality of Γ implies that $\pi \circ \pi' \models \Gamma$, and thus, $\pi \circ \pi' = \pi$, since π is a maximal model of Γ , and $\pi \circ \pi' \supseteq \pi$. Therefore, $V_{\pi} \cup V_{\pi'} = V_{\pi}$, and hence, $V_{\pi'} \subseteq V_{\pi}$. If π' is a maximal model of Γ then the same argument implies that $V_{\pi'} \supseteq V_{\pi}$ and thus $V_{\pi'} = V_{\pi}$. Conversely, if $V_{\pi'} = V_{\pi}$ then π' is a maximal model of Γ , since otherwise π' could be extended to a maximal model π'' of Γ , which would imply $V_{\pi'} \supsetneq V_{\pi''} = V_{\pi} = V_{\pi'}$.

Assume that $\alpha \equiv_{\pi} \beta$. Then $\alpha(V_{\pi}) = \beta(V_{\pi})$, and so $\alpha(V_{\pi'}) = \beta(V_{\pi'})$ and $\pi' \models \alpha \equiv \beta$. Since π' is arbitrary, we have $\Gamma \models \alpha \equiv \beta$. $\square$

Proposition 2 justifies the next definition, and implies the following lemma.

Definition 4. *For consistent and compositional $\Gamma \subseteq \mathcal{L}$, we write V^{Γ} for the set of variables appearing in any maximal model of Γ , i.e., $V^{\Gamma} = V_{\pi}$, for any maximal model π of Γ .*

Lemma 11. *Consider any consistent and compositional $\Gamma \subseteq \mathcal{L}$, and any $\alpha, \beta \in \underline{V}$. Then $\Gamma \models \alpha \equiv \beta \iff \alpha(V^{\Gamma}) = \beta(V^{\Gamma})$.*

Proof. Assume that $\alpha(V^{\Gamma}) \neq \beta(V^{\Gamma})$. Consider any maximal model π of Γ . Then, $V_{\pi} = V^{\Gamma}$ so α and β differ on some variable involved in π , which implies that $\pi \not\models \alpha \equiv \beta$, and thus, $\Gamma \not\models \alpha \equiv \beta$.

To prove the converse, assume that $\alpha(V^{\Gamma}) = \beta(V^{\Gamma})$. Then for any $\pi \in \mathcal{G}$ with $\pi \models \Gamma$, we have $V_{\pi} \subseteq V^{\Gamma}$, by Proposition 2, and thus, $\pi \models \alpha \equiv \beta$. Therefore, $\Gamma \models \alpha \equiv \beta$. $\square$

3.5 Further Properties for Inference with (Strongly) Compositional Statements

When φ is strongly compositional then there is a further simple composition property just involving $\models^*$, as expressed by the following lemma.

Lemma 12. *Let $\varphi \in \mathcal{L}$, and let $\pi, \pi' \in \mathcal{G}$. If φ is strongly compositional then $\pi \models^* \varphi$ and $\pi' \models^* \varphi$ implies $\pi \circ \pi' \models^* \varphi$. Also, if $\Gamma \subseteq \mathcal{L}$ is strongly compositional then $\pi \models^* \Gamma$ and $\pi' \models^* \Gamma$ implies $\pi \circ \pi' \models^* \Gamma$.*

Proof. Assume that $\pi \models^* \varphi$ and $\pi' \models^* \varphi$, so there exists π'' with $\pi'' \supseteq \pi'$ and $\pi'' \models \varphi$. If φ is strongly compositional then $\pi \circ \pi'' \models \varphi$, and thus, $\pi \circ \pi' \models^* \varphi$, since $\pi \circ \pi'' \supseteq \pi \circ \pi'$. The second part follows immediately from the first. $\square$

The following result states that if Γ is strongly compositional and if two outcomes are equivalent in one maximal $\models^*$ -model of Γ then they are equivalent in all $\models^*$ -models of Γ . Also, the maximal $\models^*$ -models satisfy exactly the same elements of Γ .

Proposition 3. *Assume that $\Gamma \subseteq \mathcal{L}$ is strongly compositional. Let π, π' be maximal $\models^*$ -models of Γ . Then, the following hold.*

- (i) *For every $\varphi \in \Gamma$, $\pi \models \varphi \iff \pi' \models \varphi$.*
- (ii) *If Γ is consistent then the set of maximal models of Γ is equal to the set of maximal $\models^*$ -models of Γ .*

Proof. (i): Assume that $\pi' \models \varphi$. Since $\pi \models^* \Gamma$ we have $\pi \models^* \varphi$, and so $\pi \circ \pi' \models \varphi$, since φ is strongly compositional. Thus, $\pi \models \varphi$, since $\pi \circ \pi' = \pi$. Reversing the roles of π and π' in the argument, we have $\pi \models \varphi \iff \pi' \models \varphi$.

(ii): Suppose that Γ is consistent, and so there exists some π'' with $\pi'' \models \Gamma$. Now, $\pi \models^* \Gamma$, so $\pi \circ \pi'' \models \Gamma$, since Γ is strongly compositional. Then $\pi \circ \pi'' \models^* \Gamma$, and thus, $\pi \circ \pi'' = \pi$, since π is a maximal $\models^*$ -model of Γ . So we have $\pi \models \Gamma$. This argument also shows that $V_{\pi''} \subseteq V_{\pi}$, which implies that π is a maximal model of Γ , using Proposition 2. We have just shown that, assuming Γ is consistent, every maximal $\models^*$ -model of Γ is a maximal model of Γ . Also, by Proposition 2, every maximal model τ of Γ has $V_{\tau} = V_{\pi}$. We have, by Lemma 5(i), that $\tau \models^* \Gamma$. Let τ' be a maximal $\models^*$ -model of Γ with $\tau' \supseteq \tau$. By the previous argument τ' is also a maximal model of Γ , which implies, using Proposition 2, that $V_{\tau'} = V_{\pi} = V_{\tau}$, and so $\tau' = \tau$, and thus, τ is a maximal $\models^*$ -model of Γ , as required. $\square$

One can define a notion of max-model inference, which is characterised by the lemma below. We define $\pi \models^{max} \Gamma$ if π is a maximal model of Γ . We also define $\Gamma \models^{max} \varphi$ if $\pi \models \varphi$ for every maximal model π of Γ .

Lemma 13. *Let $\Gamma \cup \{\varphi, \neg\varphi\} \subseteq \mathcal{L}$ and suppose that $\Gamma \cup \{\neg\varphi\}$ is compositional. If $\Gamma \models \varphi$ then $\Gamma \models^{max} \varphi$. Now suppose that $\Gamma \not\models \varphi$ and let π be any maximal model of $\Gamma \cup \{\neg\varphi\}$. Then, $\Gamma \models^{max} \varphi \iff \pi \not\models^{max} \Gamma$, which holds if and only if $V^{\Gamma \cup \{\neg\varphi\}} \neq V^{\Gamma}$.*

Proof. The definitions immediately imply that if $\Gamma \models \varphi$ then $\Gamma \models^{max} \varphi$. Now assume that $\Gamma \not\models \varphi$, and so $\Gamma \cup \{\neg\varphi\}$ is consistent, and let π be any maximal model of $\Gamma \cup \{\neg\varphi\}$. Suppose that $\Gamma \models^{max} \varphi$. The fact that $\pi \not\models \varphi$ implies that $\pi \not\models^{max} \Gamma$. Conversely, assume that $\Gamma \not\models^{max} \varphi$ so there exists $\pi' \in \mathcal{G}$ such that $\pi' \models^{max} \Gamma$ and $\pi' \models \neg\varphi$, and thus $\pi' \models \Gamma \cup \{\neg\varphi\}$. Proposition 2 implies that $V_{\pi'} \subseteq V_{\pi}$. But since $\pi \models \Gamma$ and $\pi' \models^{max} \Gamma$ we have also $V_{\pi} \subseteq V_{\pi'}$, and thus $V_{\pi'} = V_{\pi}$. Proposition 2 then implies that $\pi \models^{max} \Gamma$.

We have $V^{\Gamma \cup \{\neg\varphi\}} = V_{\pi} \subseteq V^{\Gamma}$. Proposition 2 implies that $\pi \models^{max} \Gamma$ if and only if $V_{\pi} = V^{\Gamma}$, which is if and only if

$$V^{\Gamma \cup \{\neg\varphi\}} = V^{\Gamma}. \quad \square$$

3.6 Decreasing Preference Statements and Sets of Models

In this section we show some results that are useful in proving that certain preference statements are strongly compositional.

Let us say that $\varphi \in \mathcal{L}$ is *decreasing* if for all $\pi, \pi' \in \mathcal{G}$ with π' extending π , we have $\pi' \models \varphi \Rightarrow \pi \models \varphi$. It follows easily from the definitions that if φ is decreasing, then, for all $\pi \in \mathcal{G}$, $\pi \models^* \varphi \iff \pi \models \varphi$. This leads to the following result.

Lemma 14. *Let $\varphi \in \mathcal{L}$ be decreasing. Then φ is strongly compositional if and only if φ is compositional.*

For example, the non-strict statements of the flight connections examples from the earlier sections are decreasing, while the strict statements in these examples are not.

For $\mathcal{M} \subseteq \mathcal{G}$, we say that $\mathcal{M}$ is *decreasing* if, for any $\pi, \pi' \in \mathcal{G}$ such that π' extends π , we have $\pi' \in \mathcal{M} \Rightarrow \pi \in \mathcal{M}$.

We say that $\mathcal{M}$ *contains all models of φ* if, for all $\pi \in \mathcal{G}$, $\pi \models \varphi \Rightarrow \pi \in \mathcal{M}$.

The following result is helpful for proving that a preference statement φ is strongly compositional.

Proposition 4. *Let $\varphi \in \mathcal{L}$, and assume that φ is consistent. Let $\mathcal{M}_{\varphi}$ be a subset of $\mathcal{G}$. The following two conditions are equivalent:*

- (I) *$\mathcal{M}_{\varphi}$ is decreasing and contains all models of φ , and for all $\pi, \pi' \in \mathcal{G}$, if $\pi \in \mathcal{M}_{\varphi}$ and $\pi' \models \varphi$ then $\pi \circ \pi' \models \varphi$.*
- (II) *φ is strongly compositional, and for all $\pi \in \mathcal{G}$, $\pi \models^* \varphi \iff \pi \in \mathcal{M}_{\varphi}$.*

Proof. (I) $\Rightarrow$ (II): Assume (I). First, let us assume that $\pi \in \mathcal{M}_{\varphi}$. Since φ is consistent, there exists π' with $\pi' \models \varphi$, and so (I) implies that $\pi \circ \pi' \models \varphi$, and thus, $\pi \models^* \varphi$, since $\pi \circ \pi' \supseteq \pi$. For proving the converse let us now assume that $\pi \models^* \varphi$, so there exists $\pi' \in \mathcal{G}$ with $\pi' \supseteq \pi$ and $\pi' \models \varphi$. Thus, $\pi' \in \mathcal{M}_{\varphi}$, and because $\mathcal{M}_{\varphi}$ is decreasing, we then have $\pi \in \mathcal{M}_{\varphi}$. We have shown that for all $\pi \in \mathcal{G}$, $\pi \models^* \varphi \iff \pi \in \mathcal{M}_{\varphi}$. (I) then also implies that φ is strongly compositional.

(II) $\Rightarrow$ (I): Lemma 5(i) implies that $\mathcal{M}_{\varphi}$ contains all models of φ , and Lemma 5(ii) implies that $\mathcal{M}_{\varphi}$ is decreasing. The fact that φ is strongly compositional then implies (I). $\square$

For $\varphi, \varphi' \in \mathcal{L}$, we say that φ' is a *relaxation of φ* if $\varphi \models \varphi'$, i.e., for all $\pi \in \mathcal{G}$, $\pi \models \varphi \Rightarrow \pi \models \varphi'$.

We have the following special case of Proposition 4.

Proposition 5. *Let $\varphi, \bar{\varphi} \in \mathcal{L}$, and assume that φ is consistent. The following two conditions are equivalent:*

- (I) *$\bar{\varphi}$ is a decreasing relaxation of φ such that for all $\pi, \pi' \in \mathcal{G}$, if $\pi \models \bar{\varphi}$ and $\pi' \models \varphi$ then $\pi \circ \pi' \models \varphi$.*
- (II) *φ is strongly compositional, and for all $\pi \in \mathcal{G}$, $\pi \models^* \varphi \iff \pi \models \bar{\varphi}$.*

Proof. Define $\mathcal{M}_\varphi$ to be all $\pi \in \mathcal{G}$ such that $\pi \models \bar{\varphi}$. Note that (I) holds if and only if $\mathcal{M}_\varphi$ is decreasing and contains all models of φ , and if $\pi \in \mathcal{M}_\varphi$ and $\pi' \models \varphi$ then $\pi \circ \pi' \models \varphi$. Proposition 4 implies that (I) $\iff$ (II). $\square$

3.7 Important Instances of Strongly Compositional Preference Statements

Projections to Y

Our computational techniques can be expressed in terms of projections of preference statements to a single variable.

Let $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, let $Y \in V$ be a variable, and let $A \subseteq V - \{Y\}$ be a set of variables not containing Y . Define $\mathfrak{R}^{\downarrow Y}$, the projection of $\mathfrak{R}$ to Y , to be $\{(\alpha(Y), \beta(Y)) : (\alpha, \beta) \in \mathfrak{R}\}$. Also, define, $\mathfrak{R}_A^{\downarrow Y}$, the A -restricted projection to Y , to be the set of pairs $(\alpha(Y), \beta(Y))$ such that $(\alpha, \beta) \in \mathfrak{R}$ and $\alpha(A) = \beta(A)$. $\mathfrak{R}_A^{\downarrow Y}$ is the projection to Y of pairs that agree on A . Thus, $\mathfrak{R}^{\downarrow Y} = \mathfrak{R}_\emptyset^{\downarrow Y}$.

From [Wilson, 2014] we have (a variation of the following):

Lemma 15. *Consider any lexicographic model $\pi \in \mathcal{G}$, written as $(Y_1, \geq_1), \dots, (Y_k, \geq_k)$. where each Y_i , for $i = 1, \dots, k$, is a variable in V , and each $\geq_{Y_i}$ is a total order on $\underline{Y_i}$. For $i = 1, \dots, k$, define A_i to be the set of earlier variables than Y_i , i.e., $A_i = \{Y_1, \dots, Y_{i-1}\}$. Let $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$. Then $\succ_\pi \supseteq \mathfrak{R}$ if and only if for all $i = 1, \dots, k$, $\geq_i \supseteq \mathfrak{R}_{A_i}^{\downarrow Y_i}$.*

Note that, by Lemma 3, $\alpha \geq \beta$ is a decreasing element. Lemma 4 implies that $\alpha \geq \beta$ is compositional, and thus, strongly compositional, by Lemma 14. If $\alpha > \beta$ is inconsistent (so that $\alpha = \beta$) then it is trivially strongly compositional. Otherwise, we can use Proposition 5 with $\varphi = \alpha > \beta$ and $\bar{\varphi} = \alpha \geq \beta$ to show, using Lemma 4, that $\alpha > \beta$ is also strongly compositional and $\pi \models^* \alpha > \beta \iff \pi \models \alpha \geq \beta$. We therefore have:

Lemma 16. *For any outcomes, $\alpha, \beta \in \underline{V}$, statements $\alpha \geq \beta$ and $\alpha > \beta$ are strongly compositional, and, if $\alpha \neq \beta$, for $\pi \in \mathcal{G}$, $\pi \models^* \alpha > \beta \iff \pi \models^* \alpha \geq \beta \iff \pi \models \alpha \geq \beta$.*

Let $\mathfrak{R}$ be a subset of $\underline{V} \times \underline{V}$, and let $\varphi^\mathfrak{R}$ be some statement satisfying: $\pi \models \varphi^\mathfrak{R}$ if and only if $\succ_\pi \supseteq \mathfrak{R}$.

If π' extends π then $\succ_\pi \supseteq \succ_{\pi'}$, which implies that $\varphi^\mathfrak{R}$ is decreasing. For any lexicographic model π we have $\pi \models \varphi^\mathfrak{R}$ if and only if for all $(\alpha, \beta) \in \mathfrak{R}$, $\pi \models \alpha \geq \beta$, which implies that $\varphi^\mathfrak{R}$ is strongly compositional, by Lemmas 16 and 6. We therefore have:

Proposition 6. *For any $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, the preference statement $\varphi^\mathfrak{R}$ is strongly compositional, and, for any $\pi \in \mathcal{G}$, we have $\pi \models^* \varphi^\mathfrak{R} \iff \pi \models \varphi^\mathfrak{R}$.*

We say that ψ is a *strict version* of $\varphi^\mathfrak{R}$ if $\varphi^\mathfrak{R}$ is a relaxation of ψ , and ψ satisfies the following monotonicity property regarding the strict preferences among $\mathfrak{R}$: for all $\pi, \pi' \in \mathcal{G}$, if $\pi \models \psi$ and $\pi' \models \varphi^\mathfrak{R}$ and $\succ_{\pi'} \supseteq (\succ_\pi \cap \mathfrak{R})$ then $\pi' \models \psi$.

There are many strict versions of $\varphi^\mathfrak{R}$ (unless $\mathfrak{R}$ is very small). We give two simple examples ψ_1 and ψ_2 of strict versions of $\varphi^\mathfrak{R}$. Let ψ_1 be such that $\pi \models \psi_1$ if and only if $\succ_\pi \supseteq \mathfrak{R}$. Let ψ_2 be such that $\pi \models \psi_2$ if and only if $\succ_\pi \supseteq \mathfrak{R}$ and there exists some $(\alpha, \beta) \in \mathfrak{R}$ such that $\alpha \succ_\pi \beta$.

Proposition 7. *Let $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, and suppose that ψ is a strict version of $\varphi^\mathfrak{R}$. Then ψ is strongly compositional, and for $\pi \in \mathcal{G}$, $\pi \models^* \psi \iff \pi \models \varphi^\mathfrak{R}$.*

Proof. We have that $\varphi^\mathfrak{R}$ is a decreasing relaxation of ψ . It is sufficient to prove that for any $\pi, \pi' \in \mathcal{G}$, if $\pi \models \varphi^\mathfrak{R}$ and $\pi' \models \psi$ then $\pi \circ \pi' \models \psi$, since then we can use Proposition 5 to prove the result. Assume that $\pi \models \varphi^\mathfrak{R}$ and $\pi' \models \psi$. Since $\varphi^\mathfrak{R}$ is a relaxation of ψ , we have $\pi' \models \varphi^\mathfrak{R}$, and thus, $\pi \circ \pi' \models \varphi^\mathfrak{R}$, since $\varphi^\mathfrak{R}$ is compositional, by Proposition 6. Consider any $(\alpha, \beta) \in \mathfrak{R}$ such that $\alpha \succ_{\pi'} \beta$. We also have $\alpha \succ_\pi \beta$, because $\pi \models \varphi^\mathfrak{R}$, and so, $\alpha \succ_{\pi \circ \pi'} \beta$, using Lemma 4. We have shown that $\succ_{\pi \circ \pi'} \supseteq (\succ_{\pi'} \cap \mathfrak{R})$, which, since ψ is a strict version of $\varphi^\mathfrak{R}$, implies $\pi \circ \pi' \models \psi$, as required. $\square$

Negated Statements

Suppose that $\pi \models \neg \varphi^\mathfrak{R}$, i.e., $\pi \not\models \varphi^\mathfrak{R}$, and consider any $\pi' \in \mathcal{G}$. Then, $\pi \circ \pi'$ extends or equals π , which implies that $\pi \circ \pi' \not\models \varphi^\mathfrak{R}$, since $\varphi^\mathfrak{R}$ is decreasing. Thus, we have:

Proposition 8. *Preference statement $\neg \varphi^\mathfrak{R}$ is compositional for any $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$.*

Lemma 17. *Let $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, and let $\pi, \pi' \in \mathcal{G}$ be such that π' extends π . Suppose that for all $(\alpha, \beta) \in \mathfrak{R}$ there exists $X \in V_\pi$ such that $\alpha(X) \neq \beta(X)$. Then, for any $(\alpha, \beta) \in \mathfrak{R}$, $\alpha \succ_\pi \beta \iff \alpha \succ_{\pi'} \beta$. Also, $\pi \models \varphi^\mathfrak{R} \iff \pi' \models \varphi^\mathfrak{R}$.*

Proof. Consider any $(\alpha, \beta) \in \mathfrak{R}$. Let Y be the earliest variable in V_π such that $\alpha(Y) \neq \beta(Y)$ (this is well-defined, by the hypothesis). Then, $\alpha \succ_\pi \beta \iff \alpha \succ_{\pi'} \beta \iff \alpha(Y) >_Y \beta(Y)$, where $>_Y$ is the strict part of the ordering for Y in π . We then have $\pi \models \varphi^\mathfrak{R}$ if and only if for all $(\alpha, \beta) \in \mathfrak{R}$, $\alpha \succ_\pi \beta$ if and only if for all $(\alpha, \beta) \in \mathfrak{R}$, $\alpha \succ_{\pi'} \beta$ if and only if $\pi' \models \varphi^\mathfrak{R}$. $\square$

Lemma 18. *Let $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, and let $\pi, \pi' \in \mathcal{G}$. Suppose that for all $(\alpha, \beta) \in \mathfrak{R}$ we have $\alpha(V_\pi) = \beta(V_\pi)$. Then, $\succ_{\pi'} \cap \mathfrak{R} = \succ_{\pi \circ \pi'} \cap \mathfrak{R}$, and thus, $\pi' \models \varphi^\mathfrak{R} \iff \pi \circ \pi' \models \varphi^\mathfrak{R}$.*

Proof. Consider any $(\alpha, \beta) \in \mathfrak{R}$. We have $\alpha \equiv_\pi \beta$. By Lemma 4, $\alpha \succ_{\pi'} \beta \iff \alpha \succ_{\pi \circ \pi'} \beta$. This implies $\succ_{\pi'} \cap \mathfrak{R} = \succ_{\pi \circ \pi'} \cap \mathfrak{R}$, and $\pi' \models \varphi^\mathfrak{R} \iff \pi \circ \pi' \models \varphi^\mathfrak{R}$. $\square$

Lemma 19. *Let $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, and let $\pi \in \mathcal{G}$. Suppose that X is the first variable in π on which some pair in $\mathfrak{R}$ differs, so that there exists $(\alpha, \beta) \in \mathfrak{R}$ such that $\alpha(X) \neq \beta(X)$, and this does not hold for any earlier variable in π . If $\mathfrak{R}^{\downarrow X}$ is not acyclic then $\pi \not\models \varphi^\mathfrak{R}$.*

Proof. Let $\geq_X$ be the non-strict relation for X in π . If $\pi \models \varphi^\mathfrak{R}$ then, by Lemma 15, $\geq_X$ contains $\mathfrak{R}^{\downarrow X}$ and thus, the latter is acyclic. $\square$

Define $\mathcal{M}_{\neg \varphi^\mathfrak{R}}$ to be the set of models $\pi \in \mathcal{G}$ such that either (i) $\pi \not\models \varphi^\mathfrak{R}$ or (ii) for every variable $X \in V_\pi$, if $\mathfrak{R}^{\downarrow X}$ is acyclic then $\mathfrak{R}^{\downarrow X} \subseteq =$.

Define $\mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$ to be the set of models $\pi \in \mathcal{G}$ such that either (i) $\pi \not\models \varphi^{\mathfrak{R}}$ or (ii) there is no variable $X \in V_\pi$ such that $\mathfrak{R}^{\downarrow X}$ is acyclic and irreflexive.

Note that $\mathcal{M}_{\neg\varphi^{\mathfrak{R}}} \subseteq \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$.

Lemma 20. *If $\pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$ and $\pi' \models \neg\varphi^{\mathfrak{R}}$ then $\pi \circ \pi' \models \neg\varphi^{\mathfrak{R}}$.*

Proof. Suppose that $\pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$ and $\pi' \models \neg\varphi^{\mathfrak{R}}$. First consider the case when $\pi \not\models \varphi^{\mathfrak{R}}$. The fact that $\varphi^{\mathfrak{R}}$ is decreasing implies that $\pi \circ \pi' \not\models \varphi^{\mathfrak{R}}$, i.e., $\pi \circ \pi' \models \neg\varphi^{\mathfrak{R}}$. Now consider the other case, when $\pi \models \varphi^{\mathfrak{R}}$. If for all $(\alpha, \beta) \in \mathfrak{R}$ we have $\alpha(V_\pi) = \beta(V_\pi)$ then Lemma 18 implies that $\pi \circ \pi' \models \neg\varphi^{\mathfrak{R}}$. Otherwise, let X be the first variable in π on which some pair in $\mathfrak{R}$ differ. The definition of $\mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$ implies that $\mathfrak{R}^{\downarrow X}$ is not acyclic, and thus, $\pi \circ \pi' \models \neg\varphi^{\mathfrak{R}}$, by Lemma 19. $\square$

Lemma 21. *$\mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$ is decreasing.*

Proof. Suppose that π' extends π and that $\pi' \in \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$. We need to show that $\pi \in \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$. We proceed using proof by contradiction. Assume that $\pi \notin \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$. Thus, $\pi \models \varphi^{\mathfrak{R}}$ and there exists variable $X \in V_\pi$ such that $\mathfrak{R}^{\downarrow X}$ is acyclic and irreflexive. Since $\pi' \in \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$ and $V_{\pi'} \supseteq V_\pi$, we must have $\pi' \not\models \varphi^{\mathfrak{R}}$. Because $\mathfrak{R}^{\downarrow X}$ is irreflexive, for all $(\alpha, \beta) \in \mathfrak{R}$, α and β differ on variable X . Lemma 17 leads to the required contradiction. $\square$

Definition 5. *For $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$, we say that $\mathfrak{R}$ is simultaneously decisive if for all $X \in V$: if $\mathfrak{R}^{\downarrow X}$ is acyclic then either $\mathfrak{R}^{\downarrow X}$ is irreflexive or $\mathfrak{R}^{\downarrow X} \subseteq =$.*

Proposition 9. *If $\mathfrak{R} \subseteq \underline{V} \times \underline{V}$ is simultaneously decisive then $\neg\varphi^{\mathfrak{R}}$ is strongly compositional, and for all $\pi \in \mathcal{G}$, $\pi \models^* \neg\varphi^{\mathfrak{R}} \iff \pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$.*

Proof. We will first show that $\mathcal{M}_{\neg\varphi^{\mathfrak{R}}} = \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$. Suppose that $\pi \in \mathcal{M}'_{\neg\varphi^{\mathfrak{R}}}$. If $\pi \not\models \varphi^{\mathfrak{R}}$ then we clearly have $\pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$, so we can assume that there is no variable $X \in V_\pi$ such that $\mathfrak{R}^{\downarrow X}$ is acyclic and irreflexive. Thus, by our assumption on $\mathfrak{R}$, if $\mathfrak{R}^{\downarrow X}$ is acyclic then $\mathfrak{R}^{\downarrow X} \subseteq =$, and thus, $\pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$.

Lemma 21 then implies that $\mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$ is decreasing, and it contains all models of $\neg\varphi^{\mathfrak{R}}$. Then, Lemma 20 and Proposition 4 imply the result. $\square$

4 Preference Languages $\mathcal{L}_{pqT}$ and $\mathcal{L}'_{pqT}$

Here we show that certain relatively expressive compact preference languages are strongly compositional. This includes forms of the statements $\varphi^{\mathfrak{R}}$ from Proposition 1, where $\mathfrak{R}$ is a set of pairs of outcomes. In many natural situations, $\mathfrak{R}$ can be exponentially large; in the languages discussed here, we are able to express certain exponentially large sets $\mathfrak{R}$ compactly.

4.1 The Language $\mathcal{L}_{pqT}$

We will consider preference statements of the form $p \triangleright q \parallel T$, where $\triangleright$ is either $\geq$, or $\gg$ or $>$, and P, Q and T are subsets of V , with $(P \cup Q) \cap T = \emptyset$, and $p \in \underline{P}$ is an assignment to P , and $q \in \underline{Q}$ is an assignment to Q . The statement $p \triangleright q \parallel T$ represents that p is preferred to q if T is held constant, i.e., any outcome α extending p is preferred to any outcome β that extends q and agrees with α on variables T . Formally, the semantics of this statement relates to the set φ^* which is defined to be the set of pairs (α, β) of outcomes such that α extends p , and β extends q , and α and β agree on T , i.e., $\alpha(T) = \beta(T)$.

Statements of the form $p \geq q \parallel T$ are called *non-strict*; statements of the form $p \gg q \parallel T$, are called *fully strict*, and statements of the form $p > q \parallel T$ are called *weakly strict*.

Let $\mathcal{L}_{pqT}$ be the set of all preference statements φ of the form $p \triangleright q \parallel T$, as defined above. For any statement $\varphi \in \mathcal{L}_{pqT}$ equalling $p \triangleright q \parallel T$, we define $\varphi^{(\geq)}$ to be $p \geq q \parallel T$, the *non-strict version* of φ . For lex model π , we define:

- π satisfies $\varphi^{(\geq)}$ if $\alpha \succ_\pi \beta$ for all $(\alpha, \beta) \in \varphi^*$.
- π satisfies fully strict φ if $\alpha \succ_\pi \beta$ for all $(\alpha, \beta) \in \varphi^*$.
- π satisfies weakly strict φ if π satisfies $\varphi^{(\geq)}$ and if $\alpha \succ_\pi \beta$ for some $(\alpha, \beta) \in \varphi^*$.

For outcomes α and β , a non-strict preference of α over β can be represented as $\alpha \geq \beta \parallel \emptyset$, which is equivalent to the preference statement $\alpha \geq \beta$ introduced in Section 3, so we abbreviate it to that. Similarly, we abbreviate $\alpha > \beta \parallel \emptyset$ to $\alpha > \beta$ (which is also equivalent to $\alpha \gg \beta \parallel \emptyset$).

We can write a statement $\varphi \in \mathcal{L}_{pqT}$ as $ur \triangleright us \parallel T$, where $u \in \underline{U}$, $r \in \underline{R}$, $s \in \underline{S}$, and U, T and $R \cup S$ are (possibly empty) mutually disjoint subsets of V , and for all $X \in R \cap S$, $r(X) \neq s(X)$. For such a representation, we write $u_\varphi = u$, $r_\varphi = r$, $s_\varphi = s$, $U_\varphi = U$, $R_\varphi = R$, $S_\varphi = S$ and $T_\varphi = T$. We assume, without loss of generality, that if $|\underline{X}| = 1$ then $X \in T_\varphi$. This ensures that such a representation is unique. We also define $W_\varphi = V - (R_\varphi \cup S_\varphi \cup T_\varphi \cup U_\varphi)$.

As discussed in [Wilson, 2009] (which, however, just considers non-strict statements), this is a relatively expressive preference language. As well as preferences between outcomes, preferences between partial tuples can be expressed. *Ceteris paribus* statements can be expressed by using φ with $W_\varphi = \emptyset$. More generally, any variables in W_φ are forced to be less important than variables in R_φ and S_φ .

Projections to Y

For comparative preference statement φ we abbreviate $(\varphi^*)_A^{\downarrow Y}$ to $\varphi_A^{\downarrow Y}$.

Proposition 1 of [Wilson, 2014] leads to the following result.

Proposition 10. *Consider any element $\varphi \in \mathcal{L}_{pqT}$. Let A be a set of variables and let Y be a variable not in A . If $R_\varphi \cap S_\varphi \cap A \neq \emptyset$ then $\varphi_A^{\downarrow Y}$ is empty. Otherwise, $\varphi_A^{\downarrow Y}$ consists of all pairs $(y, y') \in \underline{Y} \times \underline{Y}$ such that (i) $y = y'$ if $Y \in T_\varphi$; (ii) $y = y' = u_\varphi(Y)$ if $Y \in U_\varphi$ (iii) $y = r_\varphi(Y)$ if $Y \in R_\varphi$; and (iv) $y' = s_\varphi(Y)$ if $Y \in S_\varphi$. Thus if $R_\varphi \cap S_\varphi \cap A = \emptyset$ and $Y \in W_\varphi$ then $\varphi_A^{\downarrow Y} = \underline{Y} \times \underline{Y}$.*

The following lemma will be used later.

Lemma 22. Consider any $\varphi \in \mathcal{L}_{pqT}$, and any set of variables $A \subseteq V$. We have the following.

- (i) There exists $(\alpha, \beta) \in \varphi^*$ such that $\alpha(A) = \beta(A)$ if and only if $R_\varphi \cap S_\varphi \cap A = \emptyset$.
- (ii) $\alpha(A) = \beta(A)$ holds for all $(\alpha, \beta) \in \varphi^*$ if and only if $A \subseteq T_\varphi \cup U_\varphi$.

Proof. (i) First suppose that $R_\varphi \cap S_\varphi \cap A \neq \emptyset$, choose some $X \in R_\varphi \cap S_\varphi \cap A$, and consider any $(\alpha, \beta) \in \varphi^*$. Then $\alpha(X) = r_\varphi(X)$ and $\beta(X) = s_\varphi(X) \neq \alpha(X)$, which shows that $\alpha(A) \neq \beta(A)$. Conversely, suppose that $R_\varphi \cap S_\varphi \cap A = \emptyset$. Let s'_φ be s_φ restricted to $S_\varphi - R_\varphi$. Let α be any outcome extending u_φ and r_φ and s'_φ . Define β by $\beta(X) = s_\varphi(X)$ if $X \in S_\varphi$, and $\beta(X) = \alpha(X)$, otherwise. Then $(\alpha, \beta) \in \varphi^*$ and $\alpha(A) = \beta(A)$, since α and β differ only on $R_\varphi \cap S_\varphi$, which is disjoint from A .

(ii) Assume first that $A \not\subseteq T_\varphi \cup U_\varphi$, i.e., $(R_\varphi \cup S_\varphi \cup W_\varphi) \cap A \neq \emptyset$. We will construct α and β such that $\alpha(A) \neq \beta(A)$ and $(\alpha, \beta) \in \varphi^*$. Let α be any outcome extending u_φ and r_φ and such that $\alpha(X) \neq s_\varphi(X)$ for all $X \in S_\varphi$. Let β be any outcome extending u_φ and s_φ and $\alpha(T_\varphi)$ and such that $\beta(X) \neq r_\varphi(X)$ for all $X \in R_\varphi$, and also $\beta(X) \neq \alpha(X)$ for all $X \in W_\varphi$ (we can do this because each element of the domain of each variable in $R_\varphi \cup S_\varphi \cup W_\varphi$ includes at least two elements). Then $(\alpha, \beta) \in \varphi^*$ and $\alpha(X) \neq \beta(X)$ for all $X \in R_\varphi \cup S_\varphi \cup W_\varphi$, which implies that $\alpha(A) \neq \beta(A)$.

To prove the converse, assume that $A \subseteq T_\varphi \cup U_\varphi$, and consider any $(\alpha, \beta) \in \varphi^*$. Then, for each $X \in T_\varphi \cup U_\varphi$, we have $\alpha(X) = \beta(X)$, and so $\alpha(A) = \beta(A)$. $\square$

Lemma 23. Let $\varphi \in \mathcal{L}_{pqT}$ and $\pi \in \mathcal{G}$. Suppose that $\pi \models \varphi^{(\geq)}$, i.e., $\succ_\pi \supseteq \varphi^*$. If $W_\varphi \cap V_\pi \neq \emptyset$ then there exists $X \in R_\varphi \cap S_\varphi \cap V_\pi$ that appears earlier in π than any variable in W_φ .

Proof. Suppose otherwise, and let X be the first variable in W_φ that appears in π , and let $\geq_X$ be the corresponding value ordering. We will define two different pairs (α, β) and (α', β') in φ^* . Let s'_φ be s_φ restricted to $S_\varphi - R_\varphi$. Let α be any outcome extending u_φ and r_φ and s'_φ . Define β by: $\beta(X)$ is an element other than $\alpha(X)$; $\beta(Y) = s_\varphi(Y)$ if $Y \in S_\varphi$; $\beta(Y) = \alpha(Y)$ for all other Y . Then $(\alpha, \beta) \in \varphi^*$, and α and β only differ on variable X and variables $R_\varphi \cap S_\varphi$. The first variable in π on which α and β differ is X , and thus, $\alpha(X) >_X \beta(X)$, since $\alpha \succ_\pi \beta$.

Now, define outcome α' which agrees with α except on X , and outcome β' which agrees with β except on X , and where $\alpha'(X) = \beta(X)$ and $\beta'(X) = \alpha(X)$. By the same argument, we have $(\alpha', \beta') \in \varphi^*$ and $\alpha'(X) >_X \beta'(X)$, i.e., $\beta(X) >_X \alpha(X)$, which is a contradiction, since $>_X$ is a total order. $\square$

The following result characterises when a lex model satisfies a non-strict preference statement in $\mathcal{L}_{pqT}$.

Proposition 11. Let φ be a non-strict element of $\mathcal{L}_{pqT}$ (so that $\varphi = \varphi^{(\geq)}$), which we write as $u_\varphi r_\varphi \geq u_\varphi s_\varphi \parallel T_\varphi$. Let

$\pi \in \mathcal{G}$ be the model $(Y_1, \geq_1), \dots, (Y_k, \geq_k)$. Let i be the smallest index such that $Y_i \in R_\varphi \cap S_\varphi$, or let $i = k + 1$ if $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$. Then, $\pi \models \varphi$ if and only if (i) $r_\varphi(Y_i) >_i s_\varphi(Y_i)$ if $R_\varphi \cap S_\varphi \cap V_\pi \neq \emptyset$, and (ii) for all $j < i$,

- (a) $Y_j \notin W_\varphi$;
- (b) if $Y_j \in R_\varphi \setminus S_\varphi$ then for all $y \in \underline{Y_j}$, $r_\varphi(Y_j) \geq_j y$, i.e., $r_\varphi(Y_j)$ is the best value of Y_j ; and
- (c) if $Y_j \in S_\varphi \setminus R_\varphi$ then for all $y \in \underline{Y_j}$, $y \geq_j s_\varphi(Y_j)$, i.e., $s_\varphi(Y_j)$ is the worst value of Y_j .

In particular, if $V_\pi \subseteq T_\varphi \cup U_\varphi$ then $\pi \models \varphi$.

This proposition can be seen to follow from the next proposition, which is an alternative version of the result.

Proposition 12. Let $\pi \in \mathcal{G}$ and φ be a non-strict element of $\mathcal{L}_{pqT}$, so that $\varphi = \varphi^{(\geq)}$. Let us say that $X \in V_\pi$ is definite if $X \in (R_\varphi \cap S_\varphi) \cup W_\varphi$, and that X is relevant if $X \in R_\varphi \cup S_\varphi \cup W_\varphi$ and there is no earlier definite variable in V_π . Thus, the set of relevant variables consists of the earliest definite variable (if there is one), plus all earlier variables not in T_φ or U_φ . As usual, we let $\geq_X$ be the total ordering associated with X in π . Then, $\pi \models \varphi$ if and only if for all relevant variables X ,

- (a) $X \notin W_\varphi$;
- (b) if $X \in R_\varphi \cap S_\varphi$ then $r_\varphi(X) >_X s_\varphi(X)$;
- (c) if $X \in R_\varphi \setminus S_\varphi$ then for all $x \in \underline{X}$, $r_\varphi(X) \geq_X x$, i.e., $r_\varphi(X)$ is the best value of X ; and
- (d) if $X \in S_\varphi \setminus R_\varphi$ then for all $x \in \underline{X}$, $x \geq_X s_\varphi(X)$, i.e., $s_\varphi(X)$ is the worst value of X .

In particular, if $V_\pi \subseteq T_\varphi \cup U_\varphi$ then $\pi \models \varphi$.

Proof. First let us assume that $\pi \models \varphi$, i.e., $\pi \models \varphi^{(\geq)}$, and thus, $\succ_\pi \supseteq \varphi^*$. We will prove that (a), (b), (c) and (d) hold.

We first define a pair (α_0, β_0) in φ^* . Let s'_φ be s_φ restricted to $S_\varphi - R_\varphi$. Let α_0 be any outcome extending u_φ and r_φ and s'_φ . Define β_0 by: $\beta_0(Y) = s_\varphi(Y)$ if $Y \in S_\varphi$; $\beta_0(Y) = \alpha_0(Y)$ for all other Y . The only variables on which α_0 and β_0 differ are those in $R_\varphi \cap S_\varphi$, and we have $\alpha_0(R_\varphi) = r_\varphi$ and $\beta_0(S_\varphi) = s_\varphi$. We then have $(\alpha_0, \beta_0) \in \varphi^*$.

(a): Suppose that $X \in W_\varphi$. Let x be any element of $\underline{X}$ other than $\alpha_0(X)$, and let $x' = \alpha_0(X)$. Define β_1 by $\beta_1(X) = x$, and for all other $Y \in V \setminus \{X\}$, $\beta_1(Y) = \beta_0(Y)$. Also, define α_1 by $\alpha_1(X) = x$, and for all $Y \in V \setminus \{X\}$, $\alpha_1(Y) = \alpha_0(Y)$. It follows that (α_0, β_1) and (α_1, β_0) are in φ^* , and thus, $\alpha_0 \succ_\pi \beta_1$ and $\alpha_1 \succ_\pi \beta_0$, because $\succ_\pi \supseteq \varphi^*$. The first variable in π on which α_0 and β_1 differ is X , and thus, $\alpha_0(X) >_X \beta_1(X)$, i.e., $x' >_X x$. Similarly, the first variable in π on which α_1 and β_0 differ is X , and thus, $\alpha_1(X) >_X \beta_0(X)$, i.e., $x >_X x'$, contradicting the fact that $\geq_X$ is a total order.

(b): Assume that $X \in R_\varphi \cap S_\varphi$, and so $\alpha_0(X) \neq \beta_0(X)$. Since $(\alpha_0, \beta_0) \in \varphi^*$ we have $\alpha_0 \succ_\pi \beta_0$. Let Y be the first variable on which α_0 and β_0 differ, so $Y \in R_\varphi \cap S_\varphi$. Y is thus a definite variable. Since X is relevant, there is no earlier definite variable, so $Y = X$, and X is the first variable on which α_0 and β_0 differ. $\alpha_0 \succ_\pi \beta_0$ implies that $\alpha_0(X) >_X \beta_0(X)$, i.e., $r_\varphi(X) >_X s_\varphi(X)$, proving (b).

(c): Assume that $X \in R_\varphi \setminus S_\varphi$ and $\pi \models \varphi$. Choose any $x \in \underline{X}$. Let β_2 be an outcome that only differs with β_0 on X , and with $\beta_2(X) = x$. Then, $(\alpha_0, \beta_2) \in \varphi^*$, and so $\alpha_0 \succ_\pi \beta_2$, since $\pi \models \varphi$. Now, α_0 and β_2 do not differ on any earlier variables, since no earlier variable is in $R_\varphi \cap S_\varphi$, because X is relevant. This implies that $\alpha_0(X) \geq_X \beta_2(X)$, i.e., $r_\varphi(X) \geq_X x$.

(d): Assume that $X \in S_\varphi \setminus R_\varphi$ and $\pi \models \varphi$. The proof of (d) is analogous to that of (c). Choose any $x \in \underline{X}$. Let α_2 be an outcome that only differs with α_0 on X , and with $\alpha_2(X) = x$. Then, $(\alpha_2, \beta_0) \in \varphi^*$, and so, $\alpha_2(X) \geq_X \beta_0(X)$, i.e., $x \geq_X s_\varphi(X)$.

To prove the converse, we now assume that for all relevant variables, conditions (a), (b), (c) and (d) hold. We will prove that $\pi \models \varphi$. It is sufficient to show that for all $(\alpha, \beta) \in \varphi^*$ we have $\alpha \succ_\pi \beta$. So, consider any $(\alpha, \beta) \in \varphi^*$. If $\alpha(V_\pi) = \beta(V_\pi)$ then we have $\alpha \succ_\pi \beta$, so we can assume that α and β differ on some variable in V_π ; let X be the first such variable, and let A be the set of earlier variables, so that $\alpha(A) = \beta(A)$. By the definition of a lexicographic order, to prove that $\alpha \succ_\pi \beta$, it is sufficient to prove that $\alpha(X) \geq_X \beta(X)$ (i.e., $\alpha(X) >_X \beta(X)$, since $\alpha(X) \neq \beta(X)$).

Suppose that there exists a definite variable, and let Y be the earliest (according, as always to the V_π ordering). Then Y is relevant. By condition (a), $Y \notin W_\varphi$ and so $Y \in R_\varphi \cap S_\varphi$, but then $\alpha(Y) = r_\varphi(Y) \neq s_\varphi(Y) = \beta(Y)$, so $\alpha(Y) \neq \beta(Y)$. In particular this implies that $Y \notin A$, so A contains no definite variable. Since $\alpha(X) \neq \beta(X)$, we have $X \notin T_\varphi \cup U_\varphi$, so X is relevant.

If $X \in R_\varphi \cap S_\varphi$ then the definition of φ^* implies that $\alpha(X) = r_\varphi(X)$ and $\beta(X) = s_\varphi(X)$, and thus, $\alpha(X) >_X \beta(X)$, by condition (b). If $X \in R_\varphi \setminus S_\varphi$ then $\alpha(X) = r_\varphi(X)$ and condition (c) implies that $\alpha(X) \geq_X \beta(X)$. Similarly, if $X \in S_\varphi \setminus R_\varphi$ then condition (d) implies that $\alpha(X) \geq_X \beta(X)$. $\square$

The next result gives the extra conditions required for satisfying strict statements.

Proposition 13. *Let $\varphi \in \mathcal{L}_{pqT}$ and $\pi \in \mathcal{G}$. Then:*

- if φ is a fully strict statement then $\pi \models \varphi$ if and only if $\pi \models \varphi^{(\geq)}$ and $R_\varphi \cap S_\varphi \cap V_\pi \neq \emptyset$;
- if φ is a weakly strict statement then $\pi \models \varphi$ if and only if $\pi \models \varphi^{(\geq)}$ and $(R_\varphi \cup S_\varphi) \cap V_\pi \neq \emptyset$.

Proof. The definitions immediately imply that if $\pi \models \varphi$ then $\pi \models \varphi^{(\geq)}$, so we can assume that in all cases $\pi \models \varphi^{(\geq)}$.

Suppose that φ is a fully strict statement. Now $\pi \models \varphi^{(\geq)}$ implies that $\alpha \succ_\pi \beta$ for all $(\alpha, \beta) \in \varphi^*$. Therefore, $\pi \models \varphi$ if and only if for all $(\alpha, \beta) \in \varphi^*$, $\alpha \not\equiv_\pi \beta$, i.e., $\alpha(V_\pi) \neq \beta(V_\pi)$. Lemma 22(i) then implies that $\pi \models \varphi$ if and only if $R_\varphi \cap S_\varphi \cap V_\pi \neq \emptyset$.

Assume now that φ is a weakly strict statement, and also assume that $\pi \models \varphi^{(\geq)}$. We then have $\pi \models \varphi$ if and only if there exists $(\alpha, \beta) \in \varphi^*$ with $\alpha(V_\pi) \neq \beta(V_\pi)$, which, by Lemma 22(ii), is if and only if $T_\varphi \cup U_\varphi \not\subseteq V_\pi$, i.e., $(R_\varphi \cup S_\varphi \cup W_\varphi) \cap V_\pi \neq \emptyset$. Now, Proposition 12 implies that if $\pi \models \varphi$ and $W_\varphi \cap V_\pi \neq \emptyset$ then $R_\varphi \cap V_\pi \neq \emptyset$, and thus,

$\pi \models \varphi$ if and only if $(R_\varphi \cup S_\varphi) \cap V_\pi \neq \emptyset$. $\square$

Lemma 24. *Suppose $\varphi \in \mathcal{L}_{pqT}$ is such that $R_\varphi = S_\varphi$. Then φ^* is simultaneously decisive (see Definition 5).*

Proof. Let $\mathfrak{R} = \varphi^*$ and consider any $X \in V$ such that $\mathfrak{R}^{\downarrow X}$ is acyclic and $\mathfrak{R}^{\downarrow X} \not\subseteq =$. Proposition 10 implies that $X \in R_\varphi \cup S_\varphi = R_\varphi = S_\varphi$, and so $\mathfrak{R}^{\downarrow X}$ equals $\{(r_\varphi(X), s_\varphi(X))\}$, which is irreflexive since $r_\varphi(X) \neq s_\varphi(X)$. $\square$

As one would expect, both kinds of strict statements are strict versions of $\varphi^{\mathfrak{R}}$ (with a fully strict statement corresponding to ψ_1 , and a weakly strict statement corresponding to ψ_2 , in the discussion before Proposition 7).

Lemma 25. *Suppose that $\varphi \in \mathcal{L}_{pqT}$ is either a fully strict statement or a weakly strict statement. Then φ is a strict version of $\varphi^{\mathfrak{R}}$, where $\mathfrak{R} = \varphi^*$.*

Proposition 1 can be seen to imply that the non-strict elements of the language $\mathcal{L}_{pqT}$ are strongly compositional. In fact this also holds for both kinds of strict statements and certain negations.

Theorem 2. *Consider any $\varphi \in \mathcal{L}_{pqT}$. Then φ is strongly compositional and $\pi \models^* \varphi$ if and only if $\pi \models \varphi^{(\geq)}$. If φ is non-strict then $\neg\varphi$ is compositional; if also, $R_\varphi = S_\varphi$ then $\neg\varphi$ is strongly compositional, and $\pi \models^* \neg\varphi$ if and only if either $\pi \models \neg\varphi$ or $V_\pi \cap S_\varphi = \emptyset$.*

Proof. Let $\mathfrak{R} = \varphi^*$. First suppose that φ is either a fully strict statement or a weakly strict statement. For all $\pi \in \mathcal{G}$ we have $\pi \models \varphi^{(\geq)} \iff \pi \models \varphi^{\mathfrak{R}}$. Then, by Lemma 25, φ is a strict version of $\varphi^{\mathfrak{R}}$. Proposition 7 implies that φ is strongly compositional and, for $\pi \in \mathcal{G}$, $\pi \models^* \varphi$ if and only if $\pi \models \varphi^{\mathfrak{R}}$, which is if and only if $\pi \models \varphi^{(\geq)}$.

Now suppose that φ is non-strict. Then for all $\pi \in \mathcal{G}$ we have $\pi \models \varphi \iff \pi \models \varphi^{\mathfrak{R}}$, and thus also, $\pi \models^* \varphi \iff \pi \models^* \varphi^{\mathfrak{R}}$. Proposition 6 implies that $\varphi^{\mathfrak{R}}$ is strongly compositional, and $\pi \models^* \varphi^{\mathfrak{R}} \iff \pi \models \varphi^{\mathfrak{R}}$ for any $\pi \in \mathcal{G}$. Thus, φ is strongly compositional, and, for all $\pi \in \mathcal{G}$ we have $\pi \models^* \varphi \iff \pi \models \varphi^{(\geq)}$, since $\varphi^{(\geq)} = \varphi$. Proposition 8 implies that $\neg\varphi$ is compositional.

Now, assume also that $R_\varphi = S_\varphi$. Then $\mathfrak{R} = \varphi^*$ is simultaneously decisive, by Lemma 24. Proposition 9 implies that $\neg\varphi^{\mathfrak{R}}$ and thus $\neg\varphi$ is strongly compositional and for all $\pi \in \mathcal{G}$, $\pi \models^* \neg\varphi \iff \pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}}$.

We have: $\pi \in \mathcal{M}_{\neg\varphi^{\mathfrak{R}}} \iff$ either (i) $\pi \not\models \varphi^{\mathfrak{R}}$ or (ii) for every variable $X \in V_\pi$, either $\mathfrak{R}^{\downarrow X}$ is not acyclic or $\mathfrak{R}^{\downarrow X} \subseteq =$. (ii) holds if and only if for every $X \in V_\pi$, we have $X \notin R_\varphi \cup S_\varphi$ (i.e., $X \notin S_\varphi$, since $R_\varphi = S_\varphi$), so (ii) holds if and only if $V_\pi \cap S_\varphi = \emptyset$. Thus, $\pi \models^* \neg\varphi$ holds if and only if either $\pi \models \neg\varphi$ or $V_\pi \cap S_\varphi = \emptyset$. $\square$

4.2 Checking Consistency for Subsets of $\mathcal{L}'_{pqT}$

Theorem 2 suggests the feasibility of checking consistency of subsets of the language $\mathcal{L}'_{pqT}$, which is $\mathcal{L}_{pqT}$

with certain negated statements also included. Formally, define $\mathcal{L}'_{pqT}$ to be the union of $\mathcal{L}_{pqT}$ with $\{\neg\varphi : \varphi \in \mathcal{L}_{pqT}, \varphi \text{ non-strict, and } R_\varphi = S_\varphi\}$.

We use the method of Section 3.3 to determine the consistency of a set of preference statements $\Gamma \subseteq \mathcal{L}'_{pqT}$, by incrementally extending a maximal $\models^*$ -model π of Γ , and then checking whether or not $\pi \models \Gamma$ holds; this makes use of Propositions 11 and 13.

Let $\Gamma \subseteq \mathcal{L}'_{pqT}$, let $X \in V$ and let $\pi \in \mathcal{G}$. We make the following definitions, where $\bar{\Gamma}$ is the set of all $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ such that $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$.

- $\text{Best}_\Gamma^\pi(X) = \{r_\varphi(X) : \varphi \in \bar{\Gamma} \text{ \& } X \in R_\varphi \setminus S_\varphi\}$.
- $\text{Worst}_\Gamma^\pi(X) = \{s_\varphi(X) : \varphi \in \bar{\Gamma} \text{ \& } X \in S_\varphi \setminus R_\varphi\}$.
- $\text{Pairs}_\Gamma^\pi(X) = \text{Pos}_\Gamma^\pi(X) \cup \text{Neg}_\Gamma^\pi(X)$, where

$\text{Pos}_\Gamma^\pi(X) = \{(r_\varphi(X), s_\varphi(X)) : \varphi \in \bar{\Gamma} \text{ \& } X \in R_\varphi \cap S_\varphi\}$; and $\text{Neg}_\Gamma^\pi(X)$ is the set of all pairs $(s_\varphi(X), r_\varphi(X))$ such that $\neg\varphi \in \Gamma$ and $T_\varphi \cup U_\varphi \supseteq V_\pi$, and $X \in R_\varphi (= S_\varphi)$.

Lemma 26. *Suppose that $\Gamma \subseteq \mathcal{L}'_{pqT}$. Let $X \in V$ and let $\geq_X$ be a total ordering on $\underline{X}$, and let $\pi' = \pi \circ (X, \geq_X)$. Suppose that $\pi' \models^* \Gamma$. Then the following hold:*

- For all $x \in \text{Best}_\Gamma^\pi(X)$ and $x' \in \underline{X}$ we have $x \geq_X x'$. In particular then, $|\text{Best}_\Gamma^\pi(X)| \leq 1$.
- For all $x \in \text{Worst}_\Gamma^\pi(X)$ and $x' \in \underline{X}$ we have $x' \geq_X x$. In particular then, $|\text{Worst}_\Gamma^\pi(X)| \leq 1$.
- If $(x, x') \in \text{Pairs}_\Gamma^\pi(X)$ then $x \geq_X x'$.

Proof. Since $\pi' \models^* \Gamma$ we have, by Theorem 2, $\pi' \models \varphi(\geq)$ for $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$, and for $\neg\varphi \in \Gamma$, either $\pi' \models \neg\varphi$ or $V_{\pi'} \cap S_\varphi = \emptyset$. Recall the definitions of definite and relevant variables in Proposition 12. Given $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$, we have that if φ is such that $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$ and $X \in R_\varphi \cup S_\varphi$ then X is relevant given $\varphi(\geq)$ and π' . This is because X would only not be relevant if there were an earlier definite variable Y in π' and thus in V_π ; we'd then have $Y \notin W_\varphi$, by Proposition 12 and so $Y \in R_\varphi \cap S_\varphi$, which contradicts $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$.

Suppose that $x \in \text{Best}_\Gamma^\pi(X)$. By definition, there exists $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ such that $x = r_\varphi(X)$ and $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$ and $X \in R_\varphi \setminus S_\varphi$. Since X is relevant given $\varphi(\geq)$ and π' , Proposition 12 implies that for all $x' \in \underline{X}$, $x \geq_X x'$. Since $\geq_X$ is a total order, there can be at most one element x in $\text{Best}_\Gamma^\pi(X)$. A similar argument shows that if $x \in \text{Worst}_\Gamma^\pi(X)$ then for all $x' \in \underline{X}$, we have $x' \geq_X x$ which implies that $|\text{Worst}_\Gamma^\pi(X)| \leq 1$.

Suppose that $(x, x') \in \text{Pos}_\Gamma^\pi(X)$. Then, by definition, there exists $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ such that $r_\varphi(X) = x$ and $s_\varphi(X) = x'$ and $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$ and $X \in R_\varphi \cap S_\varphi$. Since X is relevant given $\varphi(\geq)$ and π' , Proposition 12 implies that $x \geq_X x'$.

Suppose that $(x, x') \in \text{Neg}_\Gamma^\pi(X)$. Then there exists $\neg\varphi \in \Gamma$ with $s_\varphi(X) = x$ and $r_\varphi(X) = x'$ and $T_\varphi \cup U_\varphi \supseteq V_\pi$ and $X \in R_\varphi = S_\varphi$. Since $V_{\pi'} \cap S_\varphi \neq \emptyset$ and $\pi' \models^* \neg\varphi$, we have $\pi' \models \neg\varphi$, i.e., $\pi' \not\models \varphi$. The condition $T_\varphi \cup U_\varphi \supseteq V_\pi$, using Proposition 12 implies that $\pi \models \varphi(\geq)$, i.e., $\pi \models \varphi$, since φ is non-strict. Also, X is relevant given $\varphi(\geq)$ and π' , so, $\pi' \not\models \varphi$ implies, using Proposition 12, that $r_\varphi(X) \not\geq_X s_\varphi(X)$, and thus, $s_\varphi(X) >_X r_\varphi(X)$ and

$x \geq_X x'$. □

Given Γ and $\pi \in \mathcal{G}$ with $\pi \models^* \Gamma$, we say that X can be chosen next if: $X \in V - V_\pi$ and

- if $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ and $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$ then $X \notin W_\varphi$;
- $\text{Pairs}_\Gamma^\pi(X)$ is acyclic;
- $|\text{Best}_\Gamma^\pi(X)| \leq 1$ and $|\text{Worst}_\Gamma^\pi(X)| \leq 1$;
- if $x \in \text{Best}_\Gamma^\pi(X)$ then x is undominated in $\text{Pairs}_\Gamma^\pi(X)$, i.e., there exists no element of the form (x', x) in $\text{Pairs}_\Gamma^\pi(X)$;
- if $x \in \text{Worst}_\Gamma^\pi(X)$ then x is not dominating in $\text{Pairs}_\Gamma^\pi(X)$, i.e., there exists no element of the form (x, x') in $\text{Pairs}_\Gamma^\pi(X)$.

We also say that $(X, \geq_X)$ is a *valid extension* of π if (i) X can be chosen next; and (ii) $\geq_X \supseteq \text{Pairs}_\Gamma^\pi(X)$; (iii) if $x \in \text{Best}_\Gamma^\pi(X)$ then x is the best element in $\underline{X}$ with respect to $\geq_X$ (so that $x \geq y$ for all $y \in \underline{X}$); and (iv) if $x' \in \text{Worst}_\Gamma^\pi(X)$ then x' is the worst element in $\underline{X}$ with respect to $\geq_X$.

We state a basic lemma that follows easily from Proposition 12.

Lemma 27. *Suppose that $\varphi \in \mathcal{L}_{pqT}$ and $\pi \in \mathcal{G}$ are such that $\pi \models \varphi$ and $V_\pi \cap R_\varphi \cap S_\varphi = \emptyset$. Then, $V_\pi \cap W_\varphi = \emptyset$.*

The following result states the conditions needed for minimally extending π to maintain the $\models^*$ -satisfaction of Γ .

Proposition 14. *Suppose that $\Gamma \subseteq \mathcal{L}'_{pqT}$, and that $\pi \models^* \Gamma$. Let X be a variable in $V - V_\pi$ and let $\pi' = \pi \circ (X, \geq_X)$, where $\geq_X$ is a total ordering on $\underline{X}$. Then $\pi' \models^* \Gamma$ if and only if $(X, \geq_X)$ is a valid extension of π .*

Proof. Since $\pi \models^* \Gamma$, Theorem 2 implies that $\pi \models \varphi(\geq)$ for $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$, and for $\neg\varphi \in \Gamma$, either $\pi \models \neg\varphi$ or $V_\pi \cap S_\varphi = \emptyset$ (since $\neg\varphi \in \mathcal{L}'_{pqT}$ implies that $\varphi \in \mathcal{L}_{pqT}$ and φ is non-strict, and $R_\varphi = S_\varphi$).

$\Leftarrow$: We will first prove that if $(X, \geq_X)$ is a valid extension of π then $\pi' \models^* \Gamma$. For $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$, we have $\pi' \models^* \varphi$ if and only if $\pi' \models \varphi(\geq)$, by Theorem 2. Also, for $\neg\varphi \in \Gamma$ we have $\pi' \models^* \neg\varphi$ if and only if either $\pi' \models \neg\varphi$ or $V_{\pi'} \cap S_\varphi = \emptyset$.

Consider any $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$. Since $\pi \models \varphi(\geq)$, it follows using Lemma 15 that $\pi' \models \varphi(\geq)$ if and only if $\geq_X \supseteq \varphi_{V_\pi}^{\downarrow X}$.

Consider any $(x, x') \in \varphi_{V_\pi}^{\downarrow X}$. We need to show that $(x, x') \in \geq_X$, i.e., that $x \geq_X x'$; this will then imply that $\geq_X \supseteq \varphi_{V_\pi}^{\downarrow X}$, and hence, $\pi' \models \varphi(\geq)$. Since $\varphi_{V_\pi}^{\downarrow X}$ is non-empty, we have, using Proposition 10, that $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$. This implies that $X \notin W_\varphi$, because $(X, \geq_X)$ is a valid extension of π . If $x = x'$ then clearly, $x \geq_X x'$. This covers the cases when $X \in T_\varphi$ and $X \in U_\varphi$ (see Proposition 10).

If $X \in R_\varphi \cap S_\varphi$, then, by Proposition 10, $\varphi_{V_\pi}^{\downarrow X} = \{(r_\varphi(X), s_\varphi(X))\}$. Thus, $(x, x') \in \text{Pos}_\Gamma^\pi(X)$, so $x \geq_X x'$.

If $X \in R_\varphi \setminus S_\varphi$ then $r_\varphi(X) \in \text{Best}_\Gamma^\pi(X)$. Proposition 10 implies that $x = r_\varphi(X)$, and thus, $x \geq_X x'$. Similarly, if $X \in S_\varphi \setminus R_\varphi$ then $s_\varphi(X) \in \text{Worst}_\Gamma^\pi(X)$. Proposition 10 implies that $x' = s_\varphi(X)$, and thus, $x \geq_X x'$. This completes the proof that, for any $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$, we have $\geq_X \supseteq \varphi_{V_\pi}^{\downarrow X}$, and hence, $\pi' \models \varphi(\geq)$, and thus, $\pi' \models^* \varphi$.

Now suppose that $\neg\varphi \in \Gamma$, and so φ is non-strict and $R_\varphi = S_\varphi$. We will show that $\pi' \models^* \neg\varphi$. Since $\pi \models^* \neg\varphi$ we have either $\pi \models \neg\varphi$ or $V_\pi \cap S_\varphi = \emptyset$. If $\pi \models \neg\varphi$, and so $\succ_{\pi'} \not\subseteq \varphi^*$, then the fact that π' extends π implies that $\succ_{\pi'} \not\subseteq \varphi^*$ (e.g., using Lemma 3), and thus, $\succ_{\pi'} \not\subseteq \varphi^*$, and therefore $\pi' \models \neg\varphi$ and $\pi' \models^* \neg\varphi$. We now thus have only to consider the case when $\pi \models \varphi$ and $V_\pi \cap S_\varphi = \emptyset$. This implies, using Lemma 27, that $V_\pi \cap W_\varphi = \emptyset$, and thus, $V_\pi \subseteq T_\varphi \cup U_\varphi$. If $X \notin S_\varphi$ then $V_{\pi'} \cap S_\varphi = \emptyset$, and so, $\pi' \models^* \neg\varphi$. Now assume that $X \in S_\varphi$. This implies that $(s_\varphi(X), r_\varphi(X)) \in \text{Neg}_\Gamma^\pi(X)$. Because $(X, \geq_X)$ is a valid extension of π , we have $s_\varphi(X) \geq_X r_\varphi(X)$, i.e., $s_\varphi(X) >_X r_\varphi(X)$, since $s_\varphi(X) \neq r_\varphi(X)$. It cannot be the case that $\pi' \models \varphi$, since then we would have $\pi' \models \varphi^{(\geq)}$ and thus, using Lemma 15, $\geq_X \supseteq \varphi_{V_\pi}^{\downarrow X}$, which implies $r_\varphi(X) \geq_X s_\varphi(X)$ using Proposition 10, contradicting $s_\varphi(X) >_X r_\varphi(X)$. We therefore have $\pi' \models \neg\varphi$, and thus, $\pi' \models^* \neg\varphi$.

$\Rightarrow$: Assume now that $\pi' \models^* \Gamma$; we will show that $(X, \geq_X)$ is a valid extension of π . We have $X \in V - V_\pi$. Since $\pi' \models^* \Gamma$ we have $\pi' \models \varphi^{(\geq)}$ for $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$, and for $\neg\varphi \in \Gamma$, either $\pi' \models \neg\varphi$ or $V_{\pi'} \cap S_\varphi = \emptyset$. For $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ we then have $\geq_X \supseteq \varphi_{V_\pi}^{\downarrow X}$, by Lemma 15. Lemma 27 implies that if $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ and $R_\varphi \cap S_\varphi \cap V_\pi = \emptyset$ then $X \notin W_\varphi$. Lemma 26 implies that there is at most one element in $\text{Best}_\Gamma^\pi(X)$ and at most one element in $\text{Worst}_\Gamma^\pi(X)$. Also if $(x, x') \in \text{Pairs}_\Gamma^\pi(X)$ then $x \geq_X x'$ and so $\geq_X$ extends $\text{Pairs}_\Gamma^\pi(X)$, and thus, $\text{Pairs}_\Gamma^\pi(X)$ is acyclic. The same lemma also implies that if $x \in \text{Best}_\Gamma^\pi(X)$ then for all $x' \in X$, $x \geq_X x'$, and thus, by the acyclicity of $\geq_X$, x is undominated in $\text{Pairs}_\Gamma^\pi(X)$. A similar argument shows that if $x \in \text{Worst}_\Gamma^\pi(X)$ then x is not dominating in $\text{Pairs}_\Gamma^\pi(X)$. This completes the proof that $(X, \geq_X)$ is a valid extension of π . $\square$

In summary, at each stage of the incremental algorithm we see if there is a variable X that can be chosen next, and if so, generate a valid extension; if not, we then have generated a maximal $\models^*$ -model π of Γ . We check consistency of Γ by determining if π satisfies Γ . Using the fact that $|\text{Pairs}_\Gamma^\pi(X)| \leq |\Gamma|$, it can be shown that the overall complexity of checking that $\Gamma (\subseteq \mathcal{L}'_{pqT})$ is consistent is $\mathcal{O}(|V|^2|\Gamma|)$.

Checking Γ -Satisfaction of Maximal $\models^*$ -Model

This section considers the following problem: Suppose that $\pi \models^* \Gamma$, where $\Gamma \subseteq \mathcal{L}'_{pqT}$. Under what conditions do we have $\pi \models \Gamma$?

Proposition 15. *Suppose that $\Gamma \subseteq \mathcal{L}'_{pqT}$ and that $\pi \models^* \Gamma$.*

- If $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ and φ is non-strict, then $\pi \models \varphi$.
- If $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ and φ is fully strict, then $\pi \models \varphi \iff R_\varphi \cap S_\varphi \cap V_\pi \neq \emptyset$.
- If $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$ and φ is weakly strict, then $\pi \models \varphi \iff (R_\varphi \cup S_\varphi) \cap V_\pi \neq \emptyset$.
- If $\neg\varphi \in \Gamma$, where φ is a non-strict element of $\mathcal{L}_{pqT}$ with $R_\varphi = S_\varphi$, we have $\pi \models \neg\varphi \iff V_\pi \not\subseteq T_\varphi \cup U_\varphi$.

Thus, $\pi \models \Gamma$ if and only if

- for all fully strict statements φ in $\Gamma \cap \mathcal{L}_{pqT}$, $R_\varphi \cap S_\varphi \cap V_\pi \neq \emptyset$;

- for all weakly strict statements φ in $\Gamma \cap \mathcal{L}_{pqT}$, $(R_\varphi \cup S_\varphi) \cap V_\pi \neq \emptyset$;
- for all $\neg\varphi \in \Gamma$, where φ is a non-strict element of $\mathcal{L}_{pqT}$ with $R_\varphi = S_\varphi$, we have $V_\pi \not\subseteq T_\varphi \cup U_\varphi$.

Proof. First consider any $\varphi \in \Gamma \cap \mathcal{L}_{pqT}$. We have that $\pi \models^* \varphi$, which implies, by Theorem 2, that $\pi \models \varphi^{(\geq)}$. Thus, if φ is non-strict then $\pi \models \varphi$, showing the first bullet point. We can then use Proposition 13 to imply the second and third bullet points.

Now consider an element of the form $\neg\varphi$ in Γ . Theorem 2 implies that either $\pi \models \neg\varphi$ or $V_\pi \cap S_\varphi = \emptyset$. Thus, if $\pi \models \varphi$ then $V_\pi \cap S_\varphi = \emptyset$, and so $V_\pi \cap W_\varphi = \emptyset$, by Lemma 27, and so $V_\pi \cap (R_\varphi \cup S_\varphi \cup W_\varphi) = \emptyset$, i.e., $V_\pi \subseteq T_\varphi \cup U_\varphi$. Conversely, if $V_\pi \subseteq T_\varphi \cup U_\varphi$ then it follows using Proposition 12 that $\succ_\pi \supseteq \varphi^*$, and hence $\pi \models \varphi^{(\geq)}$, and so, $\pi \models \varphi$, since φ is a non-strict statement. This proves the fourth bullet point.

The second half of the result follows from the first half. $\square$

5 Optimality with Respect to Sets of Compositional Preference Statements

We consider a finite set of alternatives $\mathcal{A} \subseteq V$, and we assume that we have elicited a set Γ of preference statements; we would like to find the optimal alternatives among $\mathcal{A}$. As we will see, there are several natural definitions of *optimal* [Gelain *et al.*, 2010; Wilson and O'Mahony, 2011]. We compare some of these in this section and analyse their computational cost in the context of lex models and compositional statements in the next section.

We define pre-order relation $\succsim_\Gamma$ on outcomes by $\alpha \succsim_\Gamma \beta \iff \Gamma \models \alpha \geq \beta$, and we define equivalence relation $\equiv_\Gamma$ by $\alpha \equiv_\Gamma \beta \iff \Gamma \models \alpha \equiv \beta$, i.e., if α and β are equivalent in all models of Γ . Using a similar argument to that for Theorem 1, it can be shown that, for compositional Γ , we have $\Gamma \models \alpha \equiv \beta$ holds if and only if $\alpha \equiv_\pi \beta$, where π is an arbitrary maximal model of Γ . We also define $\succ_\Gamma$ to be the strict part of $\succsim_\Gamma$, so that $\alpha \succ_\Gamma \beta$ if and only if $\alpha \succsim_\Gamma \beta$ and $\alpha \not\equiv_\Gamma \beta$. We then say that α *strictly dominates* β .

We define $\text{CSD}_\Gamma(\mathcal{A})$ ('Can Strictly Dominate') to be the set of maximal, i.e., undominated, elements of $\mathcal{A}$ w.r.t. $\succ_\Gamma$. $\alpha \in \text{CSD}_\Gamma(\mathcal{A})$ if and only if for all $\beta \in \mathcal{A}$ which are not $\equiv_\Gamma$ -equivalent to α there exists some $\pi \models \Gamma$ with $\alpha \succ_\pi \beta$.

We define $\text{O}_\pi(\mathcal{A})$ to be the subset of the alternatives that are optimal in model π , i.e., $\{\alpha \in \mathcal{A} : \forall \beta \in \mathcal{A}, \alpha \not\succsim_\pi \beta\}$. We say that $\alpha \in \mathcal{A}$ is *necessarily optimal* in $\mathcal{A}$, written $\alpha \in \text{NO}_\Gamma(\mathcal{A})$, if α is optimal in every model, i.e., if for all $\pi \in \mathcal{G}$ with $\pi \models \Gamma$ we have $\alpha \in \text{O}_\pi(\mathcal{A})$. This holds if and only if for all $\beta \in \mathcal{A}$, and for all π with $\pi \models \Gamma$ we have $\alpha \succsim_\pi \beta$.

We say that α is *possibly optimal*, written $\alpha \in \text{PO}_\Gamma(\mathcal{A})$, if α is optimal in some model of Γ , so that $\text{PO}_\Gamma(\mathcal{A}) = \bigcup_{\pi \models \Gamma} \text{O}_\pi(\mathcal{A})$. Similarly, we say that $\alpha \in \text{POM}_\Gamma(\mathcal{A})$ if α is optimal in some maximal model of Γ . Thus we have $\text{POM}_\Gamma(\mathcal{A}) = \bigcup_{\pi \models^{max} \Gamma} \text{O}_\pi(\mathcal{A})$, where $\pi \models^{max} \Gamma$ means that π is a maximal model of Γ . α is *possibly strictly optimal* in $\mathcal{A}$, written $\alpha \in \text{PSO}_\Gamma(\mathcal{A})$, if there exists some $\pi \models \Gamma$ with

$O_\pi(A) \ni \alpha$ and $\Gamma \models \alpha \equiv \beta$ for all $\beta \in O_\pi(A)$. Thus α is in $PSO_\Gamma(A)$ if there is a model of Γ in which α is optimal, and all other optimal elements are equivalent to α .

Given $\Gamma \subseteq \mathcal{L}$, we say that α is *strictly optimal* (within $\mathcal{A}$) with respect to π if α is optimal in π and any other optimal element is equivalent to α , i.e., $\alpha \in O_\pi(A)$ and $\Gamma \models \alpha \equiv \beta$ for all $\beta \in O_\pi(A)$. We write $SO_\pi^\Gamma(A)$ for the set of such elements.

For $B \subseteq \underline{V}$, we say that B are all Γ -equivalent if for all $\alpha, \beta \in B$, we have $\Gamma \models \alpha \equiv \beta$. If $O_\pi(A)$ are all Γ -equivalent then $SO_\pi^\Gamma(A) = O_\pi(A)$, otherwise, $SO_\pi^\Gamma(A) = \emptyset$. We always have that $SO_\pi^\Gamma(A)$ are all Γ -equivalent.

Let $Opt_\Gamma^A(\alpha)$ be the set of models π of Γ that make α optimal in $\mathcal{A}$, i.e., $\{\pi \models \Gamma : O_\pi(A) \ni \alpha\}$. We define $\alpha \in MPO_\Gamma(A)$ if $Opt_\Gamma^A(\alpha)$ is maximal, in the sense that there exists no $\beta \in \mathcal{A}$ with $Opt_\Gamma^A(\beta)$ a strict superset of $Opt_\Gamma^A(\alpha)$. We say that $\alpha \in MPO_\Gamma(A)$ is *maximally possibly optimal* in $\mathcal{A}$ given Γ ; this holds if and only if there is no alternative that is optimal in the same set of lex models and more.

For $\alpha \in \mathcal{A}$, let $\Delta_\alpha^A = \{\alpha \geq \beta : \beta \in \mathcal{A}\}$.

We give some basic properties.

Lemma 28. *Let $\Gamma \subseteq \mathcal{L}$ be a set of preference statements, and let π and π' be lex models, and let α be an element of set of alternatives $\mathcal{A} \subseteq \underline{V}$. Then the following all hold.*

- If π' extends π then $O_{\pi'}(A) \subseteq O_\pi(A)$.
- $\alpha \in O_\pi(A) \iff \pi \models \Delta_\alpha^A$
- $\pi \models \Gamma \cup \Delta_\alpha^A \iff \pi \in Opt_\Gamma^A(\alpha)$
- $\alpha \in PO_\Gamma(A) \iff \Gamma \cup \Delta_\alpha^A$ is consistent $\iff Opt_\Gamma^A(\alpha)$ is non-empty.
- $\Gamma \cup \Delta_\alpha^A \models \alpha \equiv \beta \iff Opt_\Gamma^A(\beta) \supseteq Opt_\Gamma^A(\alpha)$.

Proof. Assume that π' extends π . Consider any $\alpha \in O_{\pi'}(A)$, so that, for all $\beta \in \mathcal{A}$, $\alpha \succ_{\pi'} \beta$. By Lemma 3, if $\alpha \succ_{\pi'} \beta$ then $\alpha \succ_\pi \beta$. Therefore, for all $\beta \in \mathcal{A}$, $\alpha \succ_\pi \beta$, and so $\alpha \in O_\pi(A)$.

We have: $\pi \models \Delta_\alpha^A$ if and only if for all $\beta \in \mathcal{A}$, $\alpha \succ_\pi \beta$, which is if and only if $\alpha \in O_\pi(A)$.

$\pi \models \Gamma \cup \Delta_\alpha^A$ if and only if $\pi \models \Gamma$ and $\alpha \in O_\pi(A)$, which is if and only if $\pi \in Opt_\Gamma^A(\alpha)$.

$\alpha \in PO_\Gamma(A)$ if and only if there exists some π with $\pi \models \Gamma$ and $\alpha \in O_\pi(A)$. By the second bullet point, this holds if and only if there exists π with $\pi \models \Gamma \cup \Delta_\alpha^A$, i.e., $\Gamma \cup \Delta_\alpha^A$ is consistent. This is also equivalent to $Opt_\Gamma^A(\alpha)$ being non-empty by the third bullet point.

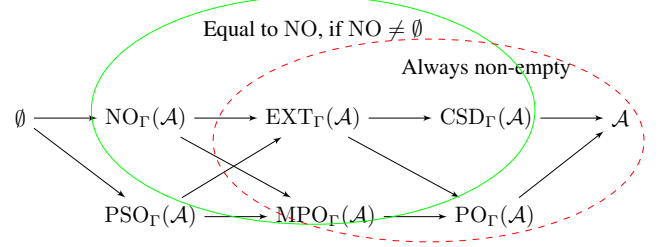
First suppose that $\Gamma \cup \Delta_\alpha^A \models \alpha \equiv \beta$, and consider any $\pi \in Opt_\Gamma^A(\alpha)$. Then, $\pi \models \Gamma \cup \Delta_\alpha^A$, and thus, $\pi \models \alpha \equiv \beta$, and so, $\alpha \equiv_\pi \beta$. This implies that $\beta \in O_\pi(A)$ and hence, $\pi \in Opt_\Gamma^A(\beta)$.

Conversely, suppose that $Opt_\Gamma^A(\beta) \supseteq Opt_\Gamma^A(\alpha)$, and consider any π such that $\pi \models \Gamma \cup \Delta_\alpha^A$; we then have $\alpha \succ_\pi \beta$. Then, by the third bullet point, $\pi \in Opt_\Gamma^A(\alpha)$, and so, $\pi \in Opt_\Gamma^A(\beta)$, which implies that $\pi \models \Gamma \cup \Delta_\beta^A$. This entails that $\beta \succ_\pi \alpha$, and thus $\alpha \equiv_\pi \beta$, i.e., $\pi \models \alpha \equiv \beta$. We have shown that $\Gamma \cup \Delta_\alpha^A \models \alpha \equiv \beta$. $\square$

Without making assumptions about Γ we have the following properties, which follow from basic arguments, that apply in a very general context [Wilson and O'Mahony, 2011] (for proofs see also [O'Mahony, 2013]).

Proposition 16. *Consider any $\mathcal{A} \subseteq \underline{V}$ and $\Gamma \subseteq \mathcal{L}$. Then, the following all hold. (i) $NO_\Gamma(A) \cup PSO_\Gamma(A) \subseteq MPO_\Gamma(A) \cap EXT_\Gamma(A)$; (ii) $EXT_\Gamma(A) \subseteq CSD_\Gamma(A) \cap PO_\Gamma(A)$; (iii) $MPO_\Gamma(A) \subseteq PO_\Gamma(A)$; (iv) $MPO_\Gamma(A) \cap EXT_\Gamma(A)$ is always non-empty. (v) If $NO_\Gamma(A)$ is non-empty then $NO_\Gamma(A) = MPO_\Gamma(A) = EXT_\Gamma(A) = CSD_\Gamma(A)$.*

We can visualise these relations in the following diagram, where $A \rightarrow B$ represents the relation $A \subseteq B$.



The following lemmas and propositions will extend these results by relations of $POM_\Gamma(A)$ and the case where Γ is compositional.

Lemma 29. *For any $\mathcal{A} \subseteq \underline{V}$ and $\Gamma \subseteq \mathcal{L}$ we have $PSO_\Gamma(A) \subseteq POM_\Gamma(A)$. If Γ is compositional then $PSO_\Gamma(A) = POM_\Gamma(A)$.*

Proof. Suppose that $\alpha \in PSO_\Gamma(A)$, so there exists $\pi \in \mathcal{G}$ with $\pi \models \Gamma$, and $O_\pi(A) \ni \alpha$, and $\Gamma \models \alpha \equiv \beta$ for all $\beta \in O_\pi(A)$. Let π' be any maximal model of Γ that extends π . Choose some β that is optimal in π' , i.e., $\beta \in O_{\pi'}(A)$. Then, using Lemma 28, $\beta \in O_\pi(A)$, and thus, $\Gamma \models \alpha \equiv \beta$, so $\pi' \models \alpha \equiv \beta$, which implies that $\alpha \in O_{\pi'}(A)$, and thus, $\alpha \in POM_\Gamma(A)$.

Assume now that Γ is compositional. Let $\alpha \in POM_\Gamma(A)$, so there exists $\pi \in \mathcal{G}$ with $\pi \models^{max} \Gamma$ and $\alpha \in O_\pi(A)$. Proposition 2 implies that for all $\beta \in O_\pi(A)$ we have $\Gamma \models \alpha \equiv \beta$, and thus, $\alpha \in PSO_\Gamma(A)$. $\square$

Lemma 30. *For any $\mathcal{A} \subseteq \underline{V}$ and compositional $\Gamma \subseteq \mathcal{L}$, we have $MPO_\Gamma(A) \subseteq POM_\Gamma(A)$.*

Proof. We will prove that $(PO_\Gamma(A) - POM_\Gamma(A)) \cap MPO_\Gamma(A) = \emptyset$. Since, $MPO_\Gamma(A) \subseteq PO_\Gamma(A)$, this implies that $MPO_\Gamma(A) \subseteq POM_\Gamma(A)$. Let $\alpha \in PO_\Gamma(A) - POM_\Gamma(A)$. By Lemma 28, $\Gamma \cup \Delta_\alpha^A$ is consistent; we choose some maximal model π of $\Gamma \cup \Delta_\alpha^A$. In particular, $O_\pi(A) \ni \alpha$. Choose some maximal model π' of Γ extending π . Then, $\alpha \notin O_{\pi'}(A)$, since $\alpha \notin POM_\Gamma(A)$. Choose some $\beta \in O_{\pi'}(A)$, and thus, $\beta \in O_\pi(A)$, by Lemma 28. This implies $\alpha \equiv_\pi \beta$, and thus, by Proposition 2, $\Gamma \cup \Delta_\alpha^A \models \alpha \equiv \beta$. This implies that $Opt_\Gamma^A(\beta) \supseteq Opt_\Gamma^A(\alpha)$, by Lemma 28. Since $\pi' \in Opt_\Gamma^A(\beta) - Opt_\Gamma^A(\alpha)$, we have that $\alpha \notin MPO_\Gamma(A)$. $\square$

A straight-forward argument implies that $PSO_\Gamma(A) \subseteq MPO_\Gamma(A)$. Lemmas 29 and 30 then imply the following.

Proposition 17. For any $\mathcal{A} \subseteq \underline{V}$ and $\Gamma \subseteq \mathcal{L}$ we have $\text{PSO}_\Gamma(\mathcal{A}) \subseteq \text{POM}_\Gamma(\mathcal{A}) \cap \text{MPO}_\Gamma(\mathcal{A})$. If Γ is compositional then $\text{PSO}_\Gamma(\mathcal{A}) = \text{POM}_\Gamma(\mathcal{A}) = \text{MPO}_\Gamma(\mathcal{A})$.

Let $\pi_1, \dots, \pi_k$ be a finite sequence of models. Define $\mathcal{A}_{\pi_1}$ to be $\text{O}_{\pi_1}(\mathcal{A})$. For $i = 1, \dots, k$ we iteratively define $\mathcal{A}_{\pi_1, \dots, \pi_i}$ to be $\text{O}_{\pi_i}(\mathcal{A}_{\pi_1, \dots, \pi_{i-1}})$. We define the *extreme elements* $\text{EXT}_\Gamma(\mathcal{A})$ as follows. $\alpha \in \text{EXT}_\Gamma(\mathcal{A})$ if and only if there exists a sequence $\pi_1, \dots, \pi_k$ of models of Γ such that $\mathcal{A}_{\pi_1, \dots, \pi_k} \ni \alpha$ and for all $\beta \in \mathcal{A}_{\pi_1, \dots, \pi_k}$, $\Gamma \models \alpha \equiv \beta$. Therefore, $\alpha \in \text{EXT}_\Gamma(\mathcal{A})$ if there is a sequence of lex models such that iteratively maximising with respect to each lex model in turn leads to a set containing α and only other alternatives that are Γ -equivalent to α .

Lemma 31. Let $\pi_1, \dots, \pi_k$ be a finite sequence of models, and let $\pi = \pi_1 \circ \dots \circ \pi_k$. Then, $\mathcal{A}_{\pi_1, \dots, \pi_k} = \mathcal{A}_\pi = \text{O}_\pi(\mathcal{A})$.

Proof. We first show that for arbitrary $\pi, \pi' \in \mathcal{G}$, $\text{O}_{\pi'}(\mathcal{A}_\pi) = \mathcal{A}_{\pi \circ \pi'}$, i.e., $\text{O}_{\pi'}(\text{O}_\pi(\mathcal{A})) = \text{O}_{\pi \circ \pi'}(\mathcal{A})$.

Consider an element $\alpha \in \text{O}_{\pi'}(\text{O}_\pi(\mathcal{A}))$; we will show that $\alpha \succ_{\pi \circ \pi'} \beta$ for every $\beta \in \mathcal{A}$, showing that $\alpha \in \text{O}_{\pi \circ \pi'}(\mathcal{A})$. We have that $\alpha \in \text{O}_\pi(\mathcal{A})$, which implies $\alpha \succ_\pi \beta$. If $\alpha \succ_\pi \beta$ then $\alpha \succ_{\pi \circ \pi'} \beta$, by Lemma 3. Otherwise, we have $\alpha \equiv_\pi \beta$, which implies that $\beta \in \text{O}_\pi(\mathcal{A})$, and thus, $\alpha \succ_{\pi'} \beta$. Lemma 4 implies that $\alpha \succ_{\pi \circ \pi'} \beta$.

Conversely, assume that $\alpha \in \text{O}_{\pi \circ \pi'}(\mathcal{A})$. Consider any $\beta \in \text{O}_\pi(\mathcal{A})$. We need to show that $\alpha \succ_{\pi'} \beta$. We have $\alpha \succ_{\pi \circ \pi'} \beta$. Lemma 3 implies that $\alpha \succ_\pi \beta$, which implies that $\alpha \in \text{O}_\pi(\mathcal{A})$, and also $\alpha \equiv_\pi \beta$. Since $\alpha \succ_{\pi \circ \pi'} \beta$, Lemma 4 implies $\alpha \succ_{\pi'} \beta$, as required.

We now prove the result by induction. It is trivial for $k = 1$. Now, $\mathcal{A}_{\pi_1, \dots, \pi_k} = \text{O}_{\pi_k}(\mathcal{A}_{\pi_1, \dots, \pi_{k-1}})$, which by the inductive hypothesis equals $\text{O}_{\pi_k}(\mathcal{A}_{\pi_1 \circ \dots \circ \pi_{k-1}})$, which equals $\mathcal{A}_{\pi_1 \circ \dots \circ \pi_k}$, by the argument above. $\square$

The optimality class $\text{EXT}_\Gamma(\mathcal{A})$ turns out also to be equivalent to $\text{PSO}_\Gamma(\mathcal{A})$ when Γ is compositional.

Proposition 18. Consider any $\mathcal{A} \subseteq \underline{V}$ and compositional $\Gamma \subseteq \mathcal{L}$. Then $\text{EXT}_\Gamma(\mathcal{A}) = \text{PSO}_\Gamma(\mathcal{A})$.

Proof. Proposition 16 implies $\text{EXT}_\Gamma(\mathcal{A}) \supseteq \text{PSO}_\Gamma(\mathcal{A})$. To prove the converse, suppose that $\alpha \in \text{EXT}_\Gamma(\mathcal{A})$. Then there exists a sequence $\pi_1, \dots, \pi_k$ of models of Γ such that $\mathcal{A}_{\pi_1, \dots, \pi_k} \ni \alpha$ and for all $\beta \in \mathcal{A}_{\pi_1, \dots, \pi_k}$, $\Gamma \models \alpha \equiv \beta$. By Lemma 31, $\alpha \in \text{O}_\pi(\mathcal{A})$, where $\pi = \pi_1 \circ \dots \circ \pi_k$, and $\Gamma \models \alpha \equiv \beta$ for all $\beta \in \text{O}_\pi(\mathcal{A})$. Since Γ is compositional, $\pi \models \Gamma$, and thus, $\alpha \in \text{PSO}_\Gamma(\mathcal{A})$. $\square$

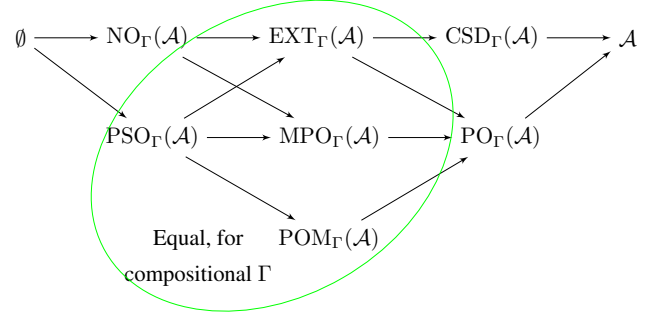
Propositions 16, 18 and 17 imply the following result, showing that there are substantial simplifications of the optimality classes when Γ is compositional.

Theorem 3. Consider any $\mathcal{A} \subseteq \underline{V}$ and compositional $\Gamma \subseteq \mathcal{L}$. Then $\text{NO}_\Gamma(\mathcal{A}) \subseteq \text{PSO}_\Gamma(\mathcal{A}) = \text{EXT}_\Gamma(\mathcal{A}) = \text{MPO}_\Gamma(\mathcal{A}) = \text{POM}_\Gamma(\mathcal{A}) \subseteq \text{CSD}_\Gamma(\mathcal{A}) \cap \text{PO}_\Gamma(\mathcal{A})$.

Proof. Proposition 17 implies that $\text{PSO}_\Gamma(\mathcal{A}) = \text{MPO}_\Gamma(\mathcal{A}) = \text{POM}_\Gamma(\mathcal{A})$. Proposition 18 implies that $\text{EXT}_\Gamma(\mathcal{A}) = \text{PSO}_\Gamma(\mathcal{A})$. Proposition 16 implies that $\text{NO}_\Gamma(\mathcal{A}) \subseteq \text{EXT}_\Gamma(\mathcal{A}) \subseteq \text{CSD}_\Gamma(\mathcal{A}) \cap \text{PO}_\Gamma(\mathcal{A})$, completing

the proof. $\square$

We can summarise the relations in the following diagram.



6 Computing Optimal Solutions

In this section, we analyse the efficiency of computing PO_Γ , PSO_Γ , CSD_Γ and NO_Γ theoretically and experimentally for $\Gamma \subseteq \mathcal{L}'_{pqT}$. Note, that by Theorem 3 since $\Gamma \subseteq \mathcal{L}'_{pqT}$ is compositional, $\text{EXT}_\Gamma = \text{MPO}_\Gamma = \text{POM}_\Gamma = \text{PSO}_\Gamma$ and thus PSO_Γ can be chosen to represent all of these classes.

Theoretical Running Times: One approach to compute $\Omega(\mathcal{A})$ for $\Omega \in \{\text{PO}_\Gamma, \text{PSO}_\Gamma, \text{CSD}_\Gamma, \text{NO}_\Gamma\}$ is to test membership for every alternative separately, i.e., checking if $\alpha \in \Omega(\mathcal{A})$ for all $\alpha \in \mathcal{A}$. By the results in Section 4.2, we can decide consistency for g statements and n variables in $\mathcal{O}(n^2g)$. To test if $\alpha \in \text{PO}_\Gamma(\mathcal{A})$, we test whether $\Gamma \cup \{\alpha \geq \beta \mid \beta \in \mathcal{A} - \{\alpha\}\}$ is consistent in $\mathcal{O}(n^2(g + m))$, where $|\mathcal{A}| = m$. Similarly, we test if $\alpha \in \text{PSO}_\Gamma(\mathcal{A})$ in $\mathcal{O}(n^2(g + m))$, by checking if $\Gamma \cup \{\alpha > \beta \mid \beta \in \mathcal{A}, \beta \not\equiv_\Gamma \alpha\}$ is consistent. To test if $\alpha \in \text{CSD}_\Gamma(\mathcal{A})$, we check for all $\beta \in \mathcal{A}$ with $\beta \not\equiv_\Gamma \alpha$ if $\Gamma \cup \{\alpha > \beta\}$ is consistent in $\mathcal{O}(mn^2g)$. To test if $\alpha \in \text{NO}_\Gamma(\mathcal{A})$, we test for all $\beta \in \mathcal{A} - \{\alpha\}$ if $\Gamma \cup \{\alpha < \beta\}$ is consistent in $\mathcal{O}(mn^2g)$. The theoretical running times for computing $\Omega(\mathcal{A})$ for $\Omega \in \{\text{PO}_\Gamma, \text{PSO}_\Gamma, \text{CSD}_\Gamma, \text{NO}_\Gamma\}$ are summarised by the following table.

	$\text{PO}_\Gamma, \text{PSO}_\Gamma$	CSD_Γ	NO_Γ
Running time	$\mathcal{O}(mn^2(g + m))$	$\mathcal{O}(m^2n^2g)$	$\mathcal{O}(m^2n^2g)$

Let $M_\Omega(m)$ denote the worst case running time of testing if an outcome $\alpha \in \mathcal{A}$ is in $\Omega(\mathcal{A})$ for $|\mathcal{A}| = m$ and operator Ω . Then the running time to compute $\Omega(\mathcal{A})$ can be estimated by $\mathcal{O}(mM_\Omega(m))$.

The work in [Wilson *et al.*, 2015b] describes the algorithm “IncrementalO” to compute $\Omega(\mathcal{A})$ for optimality operators Ω and alternatives $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ in an incremental way by testing if $\alpha_i \in \Omega(\{\alpha_1, \dots, \alpha_{i-1}\})$ and if so computing $\Omega(\{\alpha_1, \dots, \alpha_i\})$. This algorithm may be used to compute $\text{PO}_\Gamma, \text{PSO}_\Gamma$ and CSD_Γ since all of these operators satisfy path independence and thus are optimality operators.

Under the assumption that $M_\Omega(m)$ is monotonically increasing in m , we can estimate the running time $M_\Omega(m)$ of the algorithm described in [Wilson *et al.*, 2015b] to compute $\Omega(m)$ by $\mathcal{O}(\sum_{i=1, \dots, m} iM_\Omega(i))$ in the worst case and $\mathcal{O}(mM_\Omega(1))$ in the best case.

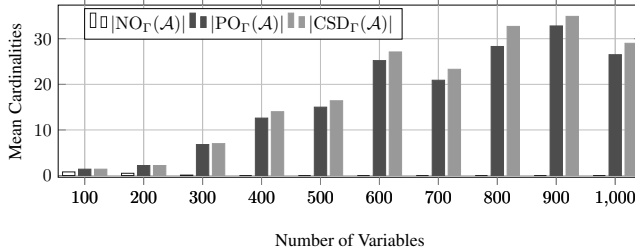
The algorithm “IncrementalO” can not be applied to compute NO_Γ , since NO_Γ does not satisfy path independence. However, a similar incremental computation is possible. Furthermore, by preprocessing the set of alternatives and including only one representative of every equivalence class w.r.t. Γ -equivalence (in $O(n^2g+mn)$), we get that $|\text{NO}_\Gamma(\mathcal{A}')| \leq 1$.

The incremental computation of computing $\Omega(\mathcal{A})$ for $\Omega \in \{\text{PO}_\Gamma, \text{PSO}_\Gamma, \text{CSD}_\Gamma, \text{NO}_\Gamma\}$ results in the following theoretical best and worst case running times.

	$\text{PO}_\Gamma, \text{PSO}_\Gamma$	CSD_Γ	NO_Γ
Best case	$\mathcal{O}(mn^2g)$	$\mathcal{O}(mn^2g)$	$\mathcal{O}(mn^2g)$
Worst case	$\mathcal{O}(m^2n^2(g+m))$	$\mathcal{O}(m^3n^2g)$	$\mathcal{O}(m^2n^2g)$

However, our experiments indicate no or only little improvement for the running times of the incremental computation compared with the previously described computation of separate membership tests. We will thus concentrate on the running time results for the computation of separate membership tests.

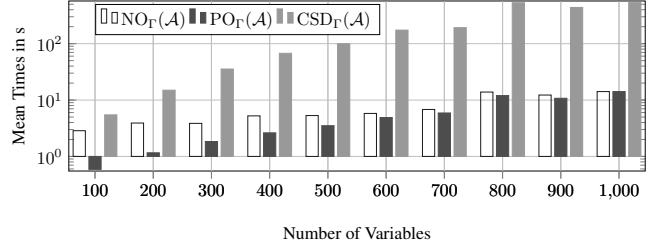
Experimental Results: We implemented the methods for computing $\Omega(\mathcal{A})$ for $\Omega \in \{\text{PO}_\Gamma, \text{PSO}_\Gamma, \text{CSD}_\Gamma, \text{NO}_\Gamma\}$ in Java Version 1.8 and conducted our experiments independently on a 2.66Ghz quad-core processor with 12GB memory. We randomly generated 10 consistent instances each, with $g \in \{100, \dots, 1000\}$ preference statements $\Gamma \subseteq \mathcal{L}'_{pqT}$ (equally divided into fully strict, weakly strict, non-strict and negated non-strict) and $n \in \{100, \dots, 1000\}$ variables with random domain sizes 2 or 3. Here, to ensure that Γ is consistent, the values of assignments involved in Γ were chosen randomly so that $\pi_S \models \Gamma$, for a given fixed lex order $\pi_S = (X_1, \geq_1) \dots (X_n, \geq_n)$. Since the experiments showed similar behavior for $m = 100, \dots, 500$ alternatives, we fix $m = 100$ in the following analysis. In every instance, the experiments result in $\text{NO}_\Gamma(\mathcal{A}) \subseteq \text{PO}_\Gamma(\mathcal{A}) = \text{PSO}_\Gamma(\mathcal{A}) \subseteq \text{CSD}_\Gamma(\mathcal{A})$ which extends the relations from Theorem 3. Here, $\text{NO}_\Gamma(\mathcal{A})$ is non-empty more often the higher the number of preference statements g and the lower the number of variables n is. The cardinality of sets $\text{PO}_\Gamma(\mathcal{A}), \text{PSO}_\Gamma(\mathcal{A}), \text{CSD}_\Gamma(\mathcal{A})$ tends to be higher the lower the number of preference statements g and the higher the number of variables n is. The diagram below shows mean cardinalities (over 10 instances each) of $\text{PO}_\Gamma(\mathcal{A}) = \text{PSO}_\Gamma(\mathcal{A}), \text{CSD}_\Gamma(\mathcal{A}), \text{NO}_\Gamma(\mathcal{A})$ for different n with $g = 1000$.



The mean running times (over 10 instances) for computing $\text{PO}_\Gamma(\mathcal{A})$ and $\text{PSO}_\Gamma(\mathcal{A})$ are similar for each instance class. They range from 0.11s to 16.5s, and tend to increase with n and g . For computing $\text{CSD}_\Gamma(\mathcal{A})$ they range from 2.69s to 659s and tend to increase with n but show no clear trend by scaling g . For computing $\text{NO}_\Gamma(\mathcal{A})$ they range from 0.09s to 18.8s and tend to increase with n and g . Thus, in all in-

stances PO , PSO , CSD and NO are computed efficiently. In 71% of the instance classes PO is computed faster than NO . Although CSD can be computed efficiently too, it is clearly computationally more expensive than PO , PSO and NO .

The following figure shows the mean running times (over 10 instances) of computing $\text{PO}_\Gamma(\mathcal{A})$, $\text{CSD}_\Gamma(\mathcal{A})$ and $\text{NO}_\Gamma(\mathcal{A})$ for different n with $g = 1000$.



Evaluation: In practice, it is often desired to compute relatively small sets of optimal solutions, e.g., not to overwhelm a user with too many choices. Since NO_Γ is usually empty, computing PSO_Γ or PO_Γ may be the best choice. This is further supported by the analysis of computational cost. In the experiments, computing PSO_Γ and PO_Γ scales much better than CSD_Γ , with the number of statements and variables, and is often faster than NO_Γ .

7 Discussion

We have developed a new approach for preference inference based on lexicographic models, using a notion of strong compositionality that allows a greedy algorithm. This is shown to be polynomial for relatively expressive preference languages. We also examined different notions of optimality, and proved relationships between them. Our experimental results confirm that the preference inference/consistency algorithm is fast, and examined the problem of generating optimal solutions, where it was found that generating the sets of possibly optimal and possibly strictly optimal solutions were significantly faster than generating the undominated solutions.

There are other common forms of preference statement that are strongly compositional, and for which the greedy algorithm will enable checking consistency. For instance, a restriction on the allowed value orderings of each variable is always strongly compositional. This can include, for instance, highly disjunctive statements, for instance, structural properties of the value orderings, such as being single-peaked [Conitzer, 2009]. Certain kinds of restrictions on variable orderings are also strongly compositional.

More formally, regarding the value ordering restrictions, for each variable $X \in V$ let $\mathcal{P}_X$ be a set of total orders on X , and let $\mathcal{P}$ be $\{\mathcal{P}_X : X \in V\}$. We say that lex model π , equalling $(Y_1, \geq_{Y_1}), \dots, (Y_k, \geq_{Y_k})$, satisfies $\mathcal{P}$ if and only if each $\geq_{Y_i}$ is in $\mathcal{P}_{Y_i}$. The definitions immediately imply that $\mathcal{P}$ is strongly compositional.

Lemma 6 showed that the property of being strongly compositional is (roughly speaking) preserved under conjunction. Although this is far from being the case for disjunctions in general, some disjunctive statements are strongly compositional; this includes the weakly strict statements in $\mathcal{L}_{pqT}$, and restrictions on value orderings.

A lexicographic order can also be viewed as a static variable and value search ordering, for a constraint satisfaction or optimisation problem. In a configuration problem, for instance, we may want to generate solutions in an ordering which fits in with what one knows about user preferences and the structure of the problem [Junker and Mailharro, 2003]. We can thus take, as input, past user preferences between solutions, or even partial tuples; or that all solutions of one constraint are preferred to all those satisfying another constraint. There may be natural restrictions on the set of value orderings, e.g., that there is a single peak or trough with a numerical domain. Given such inputs, which are strongly compositional, one can apply our approach to see if there is a compatible lexicographic order, which can then be used to generate solutions in a best first order.

Our approach in Section 3 was rather abstract, in that no explicit structure on the language $\mathcal{L}$ was assumed. In fact, a closer examination of the proofs of some of the main results suggests that these will hold more generally than for lexicographic orders: essentially they just depend on basic properties of composition and extension, but not otherwise on the structure of models. In particular, it is natural to look at the generalisation to conditional lexicographic orders, which are closely related to conditional preference languages [Wilson, 2009]. It would also be interesting to explore the potential connection with efficient classes of planning problems.

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