Preference Inference Through Rescaling Preference Learning
(Version including proofs)

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Abstract

One approach to preference learning, based on linear support vector machines, involves choosing a weight vector whose associated hyperplane has maximum margin with respect to an input set of preference vectors, and using this to compare feature vectors. However, as is well known, the result can be sensitive to how each feature is scaled, so that rescaling can lead to an essentially different vector. This gives rise to a set of possible weight vectors—which we call the rescale-optimal ones—considering all possible rescalings. From this set one can define a more cautious preference relation, in which one vector is preferred to another if it is preferred for all rescale-optimal weight vectors. In this paper, we analyse which vectors are rescale-optimal, and when there is a unique rescale-optimal vector, and we consider how to compute the induced preference relation.

1 Introduction

This document includes proofs of results in the IJCAI-2016 paper of the same name.

In many contemporary application domains, for example, information retrieval from large databases or the web, or planning in complex domains, the user has little knowledge about the set of possible solutions or feasible items, and what they generally seek is the best that's out there. But since the user does not know what is the best achievable, they typically cannot characterize it or its properties specifically [Brafman, 2008]. So, it is desirable for the system to learn the user's preferences over alternative choices (that is, documents, movies, products and so on) [Brafman and Domshlak, 2009].

Generally, a preference learning task consists of some set of items for which preferences are known, and the task is to learn a function which predicts preferences for a new set of items. An established approach to modeling preferences makes use of the concept of a *utility function*. Such a function assigns an abstract degree of utility to each alternative under consideration [Fürnkranz and Hüllermeier, 2010]. Learning a utility function from a given set of training data could be viewed from a machine learning perspective. However, training data is not necessarily given in the form of input/output pairs, but may consist of qualitative feedback in the form of pairwise comparisons, stating that one alternative is preferred to another one and therefore has a higher utility degree. Support Vector Machine (SVM) approaches [Burges, 1998] are popular in machine learning, and have inspired the development of several methods for preference learning, such as OrderSVM [Kazawa *et al.*, 2005], SVOR [Herbrich *et al.*, 1999] and SVMRank [Joachims, 2002]. Essentially, SVM-based methods are built under this assumption that the utility function is a linear weighted sum of the features. Despite the fact that a linear structure for the preference function may sound too restrictive, incorporating the *kernel* trick [Aizerman *et al.*, 1964] alleviates this by providing more flexibility, to model non-linear functions as well.

Feature spaces normalization (scaling) is an essential requirement for any SVM-based method because they are not invariant to the scale of their input feature spaces: multiplying a feature dimension by a fixed constant > 1 gives that dimension more weight in the value of the SVM objective function and, therefore, in the choice of the weight vector in the preferences function [Stolcke *et al.*, 2008; Ben-Hur and Weston, 2010]. If we base the scalings on the input instances, then it can make the induced preference relation sometimes highly sensitive to precisely which instances are received. There can thus be subjective, and even rather arbitrary, choices in the scaling of the feature spaces; different ways lead to different preference relations. This suggests defining a more cautious preference relation, consisting of all pairs that are inferred for all choices of scalings.

Thus, one alternative is preferred to another if it is preferred for all *rescale-optimal* weight vectors, where the rescale-optimal vectors are those that can be made optimal for some rescaling. This is a form of preference inference; a related form is when we only keep preferences that are supported by all compatible weight vectors, which corresponds to the kind of preference inference considered in [Marinescu *et al.*, 2013]. Other forms of preference inference, based on more qualitative, lexicographic, models are considered in [Trabelsi *et al.*, 2011; Wilson, 2014; Wilson *et al.*, 2015]. Other preference reasoning techniques based on a family of utility functions include e.g., [Greco *et al.*, 2010].

In this paper we analyse the new preference relation, deriving results regarding rescale-optimality that entail when scaling makes a difference, and that lead to a characterisation that allows computation of preference.

Section 2 defines the preference relation, and the notion of rescale-optimality, and gives some basic properties. It can happen that rescaling makes no difference; we show how to determine this in Section 3. In Sections 4 and 5 we give characterisations for rescale-optimality, that lead to a way of computing the relation, which we test out with benchmarks derived from a real ride-sharing dataset in Section 6.

2 Rescaled Maximum Margin Preference Learning

In this section we define, and give some properties of, a preference relation based on rescaling maximum margin preference learning.

2.1 Some Terminology

We first list some terminology that we'll be using throughout the paper. Consider arbitrary $u, v, w \in I\!R^n$. We say that v is non-zero if v is not equal to the zero vector $\mathbf{0} = (0, \ldots, 0) \in I\!R^n$. Also, v is said to be strictly positive if $v(j) > 0$ for all $j = 1, \ldots, n$; let $I\!R^n_+$ be the set of strictly positive elements in $I\!R^n$.

The dot product $u \cdot v$ is equal to $\sum_{j=1}^n u(j)v(j)$. The (Euclidean) norm $|u|$ of u is given by $|u|^2 = \sum_{j=1}^n (u(j))^2$, which equals $u \cdot u$. We also define $u \odot v$ to be the vector in $I\!R^n$ given by pointwise multiplication, and thus, for all $j = 1, \ldots, n$, $(u \odot v)(j) = u(j)v(j)$. Operation $\odot$ is commutative, associative and distributes over addition of vectors. An important property is that for any $u, v, w \in I\!R^n$ $(u \odot v) \cdot w = v \cdot (u \odot w)$, since they are both equal to $\sum_{j=1}^n u(j)v(j)w(j)$.

For $\Lambda \subseteq I\!R^n$, define sets Λ^*, $\Lambda^>$ and $\Lambda^{\geq 1}$ by

$$\Lambda^* = \{w \in I\!R^n : \forall \lambda \in \Lambda, w \cdot \lambda \geq 0\};$$

$$\Lambda^> = \{w \in I\!R^n : \forall \lambda \in \Lambda, w \cdot \lambda > 0\}; \text{ and}$$

$$\Lambda^{\geq 1} = \{w \in I\!R^n : \forall \lambda \in \Lambda, w \cdot \lambda \geq 1\}.$$

As is usual, we abbreviate $(\Lambda^*)^*$ to Λ^{**}. For finite $\Lambda \subseteq I\!R^n$, we define $co(\Lambda)$ to be the convex cone generated by Λ, i.e., the set of all vectors in $I\!R^n$ that can be written as $\sum_{v \in \Lambda} r_v v$, where r_v are arbitrary non-negative reals. Then $co(\Lambda)$ is finitely generated (by Λ). Elements of $co(\Lambda)$ are said to be *positive linear combinations of elements of* Λ. A *polyhedron* is the intersection of a finite number of closed half-spaces, so is topologically closed and convex and can be written as $\{w \in I\!R^n : \forall i \in I, w \cdot \lambda_i \geq a_i\}$, for finite I, where each $\lambda_i \in I\!R^n$ and $a_i \in I\!R$. If it is also a cone then it can be written with each $a_i = 0$, so that it is equal to Λ^*, where $\Lambda = \{\lambda_i : i \in I\}$. It is then called a *polyhedral cone*, and is finitely generated, by the Minkowski-Weyl Theorem for convex polyhedra.

2.2 Maximum Margin Preference Relation

We first describe a simple linear SVM-based preference relation, based on Ranking SVM [Joachims, 2002], but only considering consistent inputs. Let Λ and Θ be finite subsets of $I\!R^n$. We call Λ, the *preference inputs*, and we call Θ the *constraints*. Each input preference $\lambda \in \Lambda$ expresses a linear restriction $\lambda \cdot w > 0$ on an unknown user weight vector $w \in I\!R^n$. For instance, if there are n-features, and the user has told us that they prefer feature vector α to β (where $\alpha, \beta \in I\!R^n$, each representing the values of the alternative over the n features), then we induce from this that $\alpha \cdot w > \beta \cdot w$, i.e., $(\alpha - \beta) \cdot w > 0$, so we include $\alpha - \beta$ in Λ. (This linear weighting assumption is less restrictive than it sounds; for instance, we could form additional features representing e.g., pairwise products of the basic features, enabling a richer representation of the utility function.) The constraints set Θ is used for placing general restrictions on w; in particular, for expressing a restriction that higher values of the jth feature are at least as good; this translates to a constraint of the form $w(j) \geq 0$, represented by including e_j in Θ, where e_j is the jth unit vector, with $e_j(j) = 1$, and $e_j(k) = 0$ for $j \neq k$.

The *feasible set* $C(\Lambda, \Theta)$ is defined to be $\Lambda^> \cap \Theta^*$. We also define $G(\Lambda, \Theta)$ to be $\Lambda^{\geq 1} \cap \Theta^*$. If Θ is empty, we may abbreviate $G(\Lambda, \Theta)$ to $G(\Lambda)$ (and similarly, for other definitions).

The margin function $\text{marg}_\Lambda : C(\Lambda, \Theta) \to I\!R$ is given by $\text{marg}_\Lambda(w) = \min_{\lambda \in \Lambda} \frac{w \cdot \lambda}{|w|}$. This is equal to the distance between hyperplane $H_w = \{\mu \in I\!R^n : \mu \cdot w = 0\}$ and the closest element of Λ to H_w. The definition implies that $\text{marg}_\Lambda(w) > 0$ for all $w \in C(\Lambda, \Theta)$, since $w \cdot \lambda > 0$ for any $w \in \Lambda^>$ and $\lambda \in \Lambda$. One might view $\frac{w \cdot \lambda}{|w|}$ as the degree to which w satisfies the preference λ, with the best w being those that satisfy each λ to the greatest degree, i.e., those that maximise $\text{marg}_\Lambda(w)$.

For $w \in I\!R^n$ we define the associated relation $>_w$ by, for $\alpha, \beta \in I\!R^n$, $\alpha >_w \beta$ if and only if $w \cdot \alpha > w \cdot \beta$. Note that for any real $r > 0$, the relation $>_{rw}$ is equal to $>_w$.

Define the preference relation $>^{mm}_{\Lambda,\Theta}$ by, for $\alpha, \beta \in I\!R^n$, $\alpha >^{mm}_{\Lambda,\Theta} \beta$ if and only if there exists w with maximum margin in $C(\Lambda, \Theta)$ such that $\alpha >_w \beta$. Theorem 1 implies that $\alpha >^{mm}_{\Lambda,\Theta} \beta$ if and only if $\alpha >_{w(\Lambda,\Theta)} \beta$, where $w(\Lambda, \Theta)$ is the element in $G(\Lambda, \Theta)$ with minimum norm.[1] Thus, $>^{mm}_{\Lambda,\Theta}$ is close to being a total order, since for any $\alpha, \beta \in I\!R^n$ we have either $\alpha >^{mm}_{\Lambda,\Theta} \beta$ or $\beta >^{mm}_{\Lambda,\Theta} \alpha$ or $w(\Lambda, \Theta) \cdot (\alpha - \beta) = 0$.

Theorem 1 *For finite $\Lambda, \Theta \subseteq I\!R^n$, if $C(\Lambda, \Theta)$ is non-empty then $G(\Lambda, \Theta)$ is non-empty and there exists a unique element $w(\Lambda, \Theta)$ in $G(\Lambda, \Theta)$ with minimum norm. Also, for $w \in C(\Lambda, \Theta)$, w maximises marg_Λ within $C(\Lambda, \Theta)$ if and only if w is a strictly positive scalar multiple of $w(\Lambda, \Theta)$, i.e., there exists $r \in I\!R$ with $r > 0$ such that $w = rw(\Lambda, \Theta)$.*

This result follows from Theorem 6 in the appendix.

Example 1: Suppose that $\Lambda = \{(2, -1), (-1, 2), (\frac{1}{5}, \frac{1}{5})\}$ with Θ being empty. Then the feasible set $C(\Lambda, \Theta) = \Lambda^>$ which equals the set of all $w \in I\!R^n$ such that $2w(1) > w(2)$ and $2w(2) > w(1)$. The set $G(\Lambda, \Theta) = \Lambda^{\geq 1}$ is shown in

[1] Because of the space restrictions, not all the proofs could be included. See [Wilson and Montazery, 2016] for missing proofs.

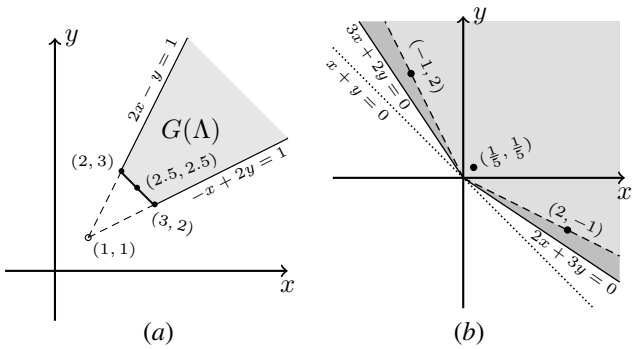


Figure 1: (a) The shaded region shows $G(\Lambda)$ when $\Lambda = \{(2,-1),(-1,2),(\frac{1}{5},\frac{1}{5})\}$, with every element of the line segment between $(3,2)$ and $(2,3)$ being rescale-optimal in $G(\Lambda)$. (b) shows the boundaries of the preferred regions for the relations $>^{mm}_{\Lambda}$, $\gg_{\Lambda}$ and $\gg^{*}_{\Lambda}$.

Figure 1(a); it has two extremal points: $(3,2)$ (corresponding to the intersection of the lines $2y-x=1$ and $x+y=5$) and $(2,3)$. The point in $G(\Lambda)$ with minimum norm is $(2.5, 2.5)$. Theorem 1 implies that the elements in $C(\Lambda,\Theta)$ with maximum margin are w with $w(1) = w(2)$. For instance, if $w = (1,1)$ then $\text{marg}_{\Lambda}(w) = \frac{1}{\sqrt{2}}\min(1,1,\frac{2}{5}) = \frac{\sqrt{2}}{5}$. The relation $>^{mm}_{\Lambda,\Theta}$ equals $>_{(2.5,2.5)}$, which is the same as $>_{(1,1)}$. Thus, $\alpha >^{mm}_{\Lambda,\Theta} \beta$, i.e., $\alpha - \beta >^{mm}_{\Lambda,\Theta} \mathbf{0}$, if and only if $(\alpha-\beta)\cdot(1,1) > 0$, i.e., $\alpha(1)+\alpha(2)-(\beta(1)+\beta(2)) > 0$. The set of γ that are $>^{mm}_{\Lambda,\Theta}$-preferred to $\mathbf{0}$ is the region in Figure 1(b) to the right of the dotted line.

2.3 The Effect of Rescaling

Consider the effect of a rescaling τ, i.e., $\tau \in I\!R^n_+$, where an element $v \in I\!R^n$ is transformed into $v \odot \tau$. An input preference vector λ becomes $\lambda \odot \tau$, so Λ becomes $\Lambda \odot \tau = \{\lambda \odot \tau : \lambda \in \Lambda\}$, and the constraints set Θ becomes $\Theta \odot \tau$. The feasible set becomes $C(\Lambda \odot \tau, \Theta \odot \tau)$. For example, if τ is the rescaling $(1,2)$ and $\Lambda = \{(2,-1),(-1,2),(\frac{1}{5},\frac{1}{5})\}$ then $\Lambda \odot \tau = \{(2,-2),(-1,4),(\frac{1}{5},\frac{2}{5})\}$. Rescaling by τ means that each $\alpha \in I\!R^n$ becomes instead $\alpha \odot \tau$. Let us say that α *is max-margin-preferred to* β *under rescaling* τ if $\alpha \odot \tau >^{mm}_{\Lambda\odot\tau,\Theta\odot\tau} \beta \odot \tau$, i.e., if rescaled α is preferred to rescaled β under the max margin relation relation corresponding to rescaled Λ and Θ. Now, it can easily happen that α is preferred to β under one rescaling, but not under another (see e.g., the example in Section 2.5). Also, the choice of how the features are scaled relative to each other can involve somewhat arbitrary choices. It is therefore natural to consider the relation given by α being preferred to β for all rescalings $\tau \in I\!R^n_+$.

Definition 1 ($\gg_{\Lambda,\Theta}$) *We define relation $\gg_{\Lambda,\Theta}$ by, for $\alpha, \beta \in I\!R^n$, $\alpha \gg_{\Lambda,\Theta} \beta$ if and only if α is max-margin-preferred to β over all rescalings, i.e., if for all $\tau \in I\!R^n_+$, $\alpha \odot \tau >^{mm}_{\Lambda\odot\tau,\Theta\odot\tau} \beta \odot \tau$.*

Let $w^*_\tau(\Lambda,\Theta)$ (abbreviated to w^*_τ) be the element with minimum norm in $G(\Lambda \odot \tau, \Theta \odot \tau)$. Theorem 1 implies that $\alpha \odot \tau >^{mm}_{\Lambda\odot\tau,\Theta\odot\tau} \beta \odot \tau$ if and only if $\alpha \odot \tau >_{w^*_\tau} \beta \odot \tau$, which can be rewritten as $\alpha >_{w^*_\tau \odot \tau} \beta$.

Defining $\text{RO}(\Lambda,\Theta)$ to be $\{w^*_\tau \odot \tau : \tau \in I\!R^n_+\}$, we have:

$$\alpha \gg_{\Lambda,\Theta} \beta \iff \text{for all } w \in \text{RO}(\Lambda,\Theta),\ \alpha >_w \beta.$$

For example, it can be shown that in Figure 1(a), $\text{RO}(\Lambda,\Theta)$ is the closed line segment between $(2,3)$ and $(3,2)$. We show below that $\text{RO}(\Lambda,\Theta)$ is equal the set of rescale-optimal elements in $G(\Lambda,\Theta)$, defined as follows.

Definition 2 (Rescale-optimal) *For $G \subseteq I\!R^n$, and $u \in G$, let us say that u* is rescale-optimal in G *if there exists strictly positive $\tau \in I\!R^n_+$ with $|\tau \odot w| \geq |\tau \odot u|$ for all $w \in G$.*

If $\mathbf{0} \in G$ then it is the unique element that is rescale-optimal in G. For $\tau \in I\!R^n_+$, we define τ^{-1} to be the element of $I\!R^n_+$ given by $\tau^{-1}(j) = 1/\tau(j)$ for all $j \in \{1,\ldots,n\}$.

Lemma 1 *Consider any $v \in I\!R^n$ and any $\tau \in I\!R^n_+$. Then, $v \in G(\Lambda,\Theta)$ if and only if $v \odot \tau \in G(\Lambda \odot \tau^{-1}, \Theta \odot \tau^{-1})$. Also, $v = w$ minimises $|w \odot \tau|$ over $w \in G(\Lambda,\Theta)$ if and only if $v = \tau^{-1} \odot w^*_{\tau^{-1}}(\Lambda,\Theta)$.*

Proof: The first part follows easily from the definitions.

Regarding the second part, by definition of $w^*_{\tau^{-1}}$, we have that $v = \tau^{-1} \odot w^*_{\tau^{-1}}(\Lambda,\Theta)$ if and only if $v \odot \tau$ has minimum norm in $G(\Lambda \odot \tau^{-1}, \Theta \odot \tau^{-1})$, which is if and only if $v = w$ minimises $|w \odot \tau|$ over $w \in G(\Lambda,\Theta)$. □

Proposition 1 $\text{RO}(\Lambda,\Theta)$ *is the set of all rescale-optimal elements of $G(\Lambda,\Theta)$. Thus, for $\alpha, \beta \in I\!R^n$, $\alpha \gg_{\Lambda,\Theta} \beta$ if and only if $\alpha >_w \beta$ for all rescale-optimal elements of $G(\Lambda,\Theta)$, which is if and only if $\alpha - \beta \in (\text{RO}(\Lambda,\Theta))^{>}$.*

Proof: Consider any $v \in I\!R^n$. Then, v is rescale-optimal in $G(\Lambda,\Theta)$ if and only if there exists $\tau \in I\!R^n_+$ such that $v = w$ minimises $|w \odot \tau|$ over $w \in G(\Lambda,\Theta)$, which, by Lemma 1, is if and only if there exists $\tau \in I\!R^n_+$ such that $v = \tau^{-1} \odot w^*_{\tau^{-1}}(\Lambda,\Theta)$, which is if and only if $v \in \text{RO}(\Lambda,\Theta)$. The last part follows immediately from the definitions. □

Another natural preference relation $\gg^*_{\Lambda,\Theta}$, which is very closely related to the one explored in [Marinescu *et al.*, 2012; 2013], is given by $\alpha \gg^*_{\Lambda,\Theta} \beta$ if and only if for all $w \in C(\Lambda,\Theta)$, $w \cdot (\alpha - \beta) > 0$, i.e., iff α is preferred to β for all compatible weight vectors. This holds if and only if for all $w \in G(\Lambda,\Theta)$, $w \cdot (\alpha - \beta) > 0$. Thus, $\alpha \gg^*_{\Lambda,\Theta} \beta$ if and only if $\alpha - \beta \in (G(\Lambda,\Theta))^{>}$. (When Θ is empty, $(G(\Lambda,\Theta))^{>}$ is equal to $co(\Lambda)$, the smallest convex cone containing Λ.) This implies that if $\alpha \gg^*_{\Lambda,\Theta} \beta$ then $\alpha \gg_{\Lambda,\Theta} \beta$. The three defined preference relations are therefore nested: $\gg^*_{\Lambda,\Theta} \subseteq \gg_{\Lambda,\Theta} \subseteq >^{mm}_{\Lambda,\Theta}$. The three relations are all irreflexive and transitive, and thus strict partial orders (with $>^{mm}_{\Lambda,\Theta}$ close to being a total order). Also, if $\succ$ is any of the three relations, $\lambda \succ \mathbf{0}$, for any $\lambda \in \Lambda$, and for $\alpha, \beta, \gamma \in I\!R^n$ and $r \in I\!R$ with $r > 0$, if $\alpha \succ \beta$ then $\alpha + \gamma \succ \beta + \gamma$ and $r\alpha > r\beta$.

2.4 Pointwise Undominated Vectors

For $u \in G(\Lambda,\Theta)$, if there exists $v \in G(\Lambda,\Theta)$ such that for all j, $v(j)$ is between $u(j)$ and 0 then it is easy to see that u is not rescale-optimal. This is the idea behind being pointwise undominated.

Definition 3 (pointwise dominance) *For $u, v \in \mathbb{R}^n$, v pointwise dominates u if $u \neq v$ and for all $j \in \{1, \ldots, n\}$, either $0 \leq v(j) \leq u(j)$ or $0 \geq v(j) \geq u(j)$. u is pointwise undominated in $G \subseteq \mathbb{R}^n$ if there exists no $v \in G$ that pointwise dominates u.*

The definition easily implies that rescale-optimality implies being pointwise undominated.

Proposition 2 *Let $G \subseteq \mathbb{R}^n$. If u is rescale-optimal in G then u is pointwise undominated in G.*

Proof: Suppose that u is not pointwise undominated in G, so that there exists $v \in G$ that pointwise dominates u. Then, for every $j \in \{1, \ldots, n\}$, $|v(j)| \leq |u(j)|$, and for some $k \in \{1, \ldots, n\}$, $|v(j)| < |u(j)|$, which implies that for every $\tau \in \mathbb{R}^n_+$, $|u \odot \tau| > |v \odot \tau|$, and hence, u is not rescale-optimal in G. □

Lemma 2 *u is pointwise undominated in convex G if and only if for all $v \in G - \{u\}$ there exists $k \in \{1, \ldots, n\}$ such that either $0 \leq u(k) < v(k)$ or $0 \geq u(k) > v(k)$.*

Proof: Suppose first that u is not pointwise undominated in G, so there exists $v \neq u$ with for all $j \in \{1, \ldots, n\}$, either $0 \leq v(j) \leq u(j)$ or $0 \geq v(j) \geq u(j)$. Then there does not exist $k \in \{1, \ldots, n\}$ with either $0 \leq u(k) < v(k)$ or $0 \geq u(k) > v(k)$.

Conversely, suppose that it is not the case that for all $v \in G - \{u\}$ there exists $k \in \{1, \ldots, n\}$ such that either $0 \leq u(k) < v(k)$ or $0 \geq u(k) > v(k)$. Then there exists $v \in G - \{u\}$ such that for all $j \in \{1, \ldots, n\}$, if $u(j) = 0$ then $v(j) = 0$; if $u(j) > 0$ then $v(j) \leq u(j)$; and if $u(j) < 0$ then $v(j) \geq u(j)$. For $\delta \in (0, 1]$ let $v_\delta = \delta v + (1 - \delta)u$, which is in G. Then there exists $\delta \in (0, 1]$ such that for all $j \in \{1, \ldots, n\}$, if $u(j) = 0$ then $v_\delta(j) = 0$; if $u(j) > 0$ then $0 < v_\delta(j) \leq u(j)$; and if $u(j) < 0$ then $0 > v_\delta(j) \geq u(j)$. Thus, v_δ pointwise dominates u showing that u is not pointwise undominated in G. □

2.5 Example 1 Continued

Since $(2.5, 2.5)$ has minimum norm in $G(\Lambda)$, it is rescale-optimal. The only pointwise undominated elements in $G(\Lambda)$ are $(3, 2)$, $(2, 3)$, and the line segment joining them, and, it turns out that all these are rescale-optimal. If we set $\tau = (r, 1)$ then $r \leq \sqrt{2/3}$ will lead to the point $(3, 2)$ having minimum value of $|(x, y) \odot \tau|$. Apart from this closed line segment, no other element of $G(\Lambda)$ can be rescale-optimal, since no other element is pointwise undominated.

Figure 1(b) illustrates the strength of the three relations, $>^{mm}_\Lambda$, $\gg_\Lambda$ and $\gg^*_\Lambda$, by showing the regions of all $\gamma \in \mathbb{R}^2$ that are preferred to $\mathbf{0}$. For $\gg^*_\Lambda$, this is the convex cone generated by the three input vectors Λ, i.e., all positive linear combinations of them. For $\gg_\Lambda$, the preferred set is the intersection of the halfspaces $\{\gamma : \gamma \cdot (3, 2) > 0\}$ and $\{\gamma : \gamma \cdot (2, 3) > 0\}$. The largest preference region is associated with $>^{mm}_\Lambda$, with $\gamma >^{mm}_{\Lambda,\Theta} \mathbf{0}$ if and only if $\gamma(x) + \gamma(y) > 0$. Since $\gg_\Lambda$ is not equal to $>^{mm}_\Lambda$, the scaling makes a difference. E.g., $(5, 0) >^{mm}_\Lambda (0, 4)$, which is the same as $(5, 0)$ being preferred to $(0, 4)$ under rescaling $(1, 1)$. However, it can be shown that $(0, 4)$ is preferred to $(5, 0)$ under rescaling $\tau = (1, 2)$, with $w^*_\tau \odot \tau$ being equal to $(2, 3)$ (which minimises $|v \odot \tau^{-1}|$ over $v \in G(\Lambda)$) and $(0, 4) \cdot (2, 3) > (5, 0) \cdot (2, 3)$.

Now consider if Λ instead equals $\{(2, -1), (-1, 2)\}$. Then $G(\Lambda)$ has a single extremal point $(1, 1)$, being the intersection of the lines $2x - y = 1$ and $2y - x = 1$. Since $(1, 1)$ is the element in $G(\Lambda)$ with minimum norm, we have $w(\Lambda, \Theta) = (1, 1)$, and $(1, 1)$ is rescale-optimal. Thus, $>^{mm}_\Lambda$ equals $>_{(1,1)}$. In fact, $(1, 1)$ pointwise dominates every other element in $G(\Lambda)$, so, by Proposition 2, is the only rescale-optimal vector in $G(\Lambda)$. Then, $\gg_\Lambda$ is just $>_{(1,1)}$, so that $\alpha \gg_\Lambda \beta$ if and only if $\alpha(x) + \alpha(y) > \beta(x) + \beta(y)$. This example shows that allowing rescaling can sometimes make no difference. In Section 3 we show a general result that u is a unique rescale-optimal element in closed convex G if and only if u pointwise dominates every other element of G.

3 Determining Uniquely Rescale-Optimal Vectors

Here we characterise the situations when rescaling makes no difference, i.e., when there is a unique rescale-optimal vector.

For convex closed $G \subseteq \mathbb{R}^n$, and for $\tau \in \mathbb{R}^n_+$ we write $w_\tau(G)$ for the unique vector $w \in G$ with minimum value of $|w \odot \tau|$, which makes sense because of the following result. Thus, u is rescale-optimal in convex closed G if and only if there exists $\tau \in \mathbb{R}^n_+$ such that $u = w_\tau(G)$.

Lemma 3 *Let G be a convex and (topologically) closed subset of $\mathbb{R}^n$. For each strictly positive vector $\tau \in \mathbb{R}^n_+$, there exists a unique $w \in G$ with minimum value of $|w \odot \tau|$.*

This follows from Proposition 4 in the appendix, since $G \odot \tau$ is convex and closed.

Proof: It is standard and well known result (see Proposition 4 in the appendix), that there is a unique element in a convex closed set with minimum norm. Consider any $\tau \in \mathbb{R}^n_+$. Now, $G \odot \tau = \{w \odot \tau : w \in G\}$ is convex and closed so there exists a unique element $w \odot \tau \in G \odot \tau$ with minimum value of $|w \odot \tau|$, so there is a unique $w \in G$ with minimum value of $|w \odot \tau|$. □

Theorem 2 below states that u is the only rescale-optimal element in convex closed G if and only if u pointwise dominates every other element of G. The proof uses a pair of lemmas.

Lemma 4 *Let $u, v \in \mathbb{R}^n$. There exists $k \in \{1, \ldots, n\}$ such that $|u(k)| < |v(k)|$ if and only if there exists $\tau \in \mathbb{R}^n_+$ such that $|u \odot \tau| < |v \odot \tau|$. Thus, for all $j \in \{1, \ldots, n\}$, $|u(j)| \geq |v(j)|$ if and only if for all $\tau \in \mathbb{R}^n_+$, $|u \odot \tau| \geq |v \odot \tau|$.*

Proof: $\Rightarrow$: Assume first that there exists $k \in \{1, \ldots, n\}$ such that $|u(k)| < |v(k)|$. For $\epsilon > 0$, define $\tau_\epsilon \in \mathbb{R}^n_+$ by $\tau_\epsilon(k) = 1 + \epsilon$, and, for $j \neq k$, $\tau_\epsilon(k) = \epsilon$. Then $u \odot \tau_\epsilon = \epsilon u + u(k)e_k$, which leads to $|u \odot \tau_\epsilon|^2 = (u \odot \tau_\epsilon) \cdot (u \odot \tau_\epsilon)$ equalling $\epsilon^2 u \cdot u + (1 + 2\epsilon)u(k)^2$. Since $u(k)^2 < v(k)^2$, for sufficiently small $\epsilon > 0$, we will have $|u \odot \tau_\epsilon|^2 < |v \odot \tau_\epsilon|^2$.

$\Leftarrow$: Now assume that there exists $\tau \in \mathbb{R}^n_+$ such that $|u \odot \tau| < |v \odot \tau|$. Then for some $k \in \{1, \ldots, n\}$, $|(u \odot \tau)(k)| < |(v \odot \tau)(k)|$, i.e., $|u(k)\tau(k)| < |v(k)\tau(k)|$, which implies $|u(k)| < |v(k)|$, since $\tau(k)$ is non-zero. □

Lemma 5 *Let G be a convex subset of $\mathbb{R}^n$, and let j be any element of $\{1, \ldots, n\}$. Then either (i) there exists $w \in G$ such that $w(j) = 0$; or (ii) for all $w \in G$, $w(j) > 0$; or (iii) for all $w \in G$, $w(j) < 0$.*

Proof: To prove a contradiction, suppose that neither (i), (ii) nor (iii) hold for j, so there exists $u, v \in G$ such that $u(j) > 0$ and $v(j) < 0$. Let $\delta = \frac{u(j)}{u(j)-v(j)}$, so that $1 - \delta = \frac{-v(j)}{u(j)-v(j)}$. Let $v_\delta = \delta v + (1 - \delta)u$, which is in G because G is convex and $\delta \in (0, 1)$. Then, $v_\delta(j) = 0$, which shows that (i) holds for j, contradicting the earlier assumption. □

Theorem 2 *Let G be a convex and closed subset of $\mathbb{R}^n$, and let u be an element of G. Then the following conditions are equivalent.*

(i) u is uniquely rescale-optimal in G, i.e., u is the unique element of G that is rescale-optimal;

(ii) for all $v \in G$, for all $j \in \{1, \ldots, n\}$, $|v(j)| \geq |u(j)|$;

(iii) u pointwise dominates every element in $G - \{u\}$.

The equivalence between (i) and (ii) is proved using Lemmas 3 and 4, and the equivalence between (ii) and (iii) follows using Lemma 5.

Proof: Recall, for $\tau \in \mathbb{R}^n_+$, that $w_\tau(G)$ is the unique vector $w \in G$ with minimum value of $|w \odot \tau|$ (see Lemma 3). First suppose (i), that u is uniquely rescale-optimal in G, so that, for all $\tau \in \mathbb{R}^n_+$, $w_\tau(G) = u$, so, for all $\tau \in \mathbb{R}^n_+$, and for all $v \in G - \{u\}$, $|u \odot \tau| \leq |v \odot \tau|$; thus, by Lemma 4, for all $j \in \{1, \ldots, n\}$, $|u(j)| \leq |v(j)|$, showing that (ii) holds. The converse follows easily: if (ii) holds then for all $\tau \in \mathbb{R}^n_+$, for all $v \in G$, $|u \odot \tau| \leq |v \odot \tau|$ and thus $u = w_\tau(G)$ proving (i).

We will next show that (ii) implies (iii). Consider any $j \in \{1, \ldots, n\}$. We must show that for each $v \in G - \{u\}$, either $0 \leq u(j) \leq v(j)$ or $0 \geq u(j) \geq v(j)$ hold. This holds trivially if $u(j) = 0$, so suppose $u(j) \neq 0$. (ii) implies that, for all $v \in G$, $v(j) \neq 0$, and thus, by Lemma 5, either, for all $v \in G$, $v(j) > 0$, or for all $v \in G$, $v(j) < 0$. (ii) then implies that for all $v \in G$, either $0 \leq u(j) \leq v(j)$ or $0 \geq u(j) \geq v(j)$, proving (iii). It immediately follows that (iii) implies (ii), completing the proof of equivalence of (i), (ii) and (iii).□

Corollary 1 *For finite $\Lambda, \Theta \subseteq \mathbb{R}^n$, let $G = G(\Lambda, \Theta)$. Define $y \in \mathbb{R}^n$ as follows. Choose an arbitrary element $z \in G$. For each $j \in \{1, \ldots, n\}$: If $z(j) = 0$ then define $y(j) = 0$. If $z(j) > 0$ then define $y(j) = \inf\{w(j) : w \in G, w(j) \geq 0\}$. If $z(j) < 0$ then define $y(j) = \sup\{w(j) : w \in G, w(j) \leq 0\}$. If $y \in G$ then y is uniquely rescale-optimal in G. Also, there exists a uniquely rescale-optimal element in G if and only if $y \in G$.*

Proof: Consider any $j \in \{1, \ldots, n\}$. Lemma 5 implies that if $y(j) > 0$ then for all $w \in G$, $0 < y(j) \leq w(j)$; and if $y(j) < 0$ then for all $w \in G$, $0 > y(j) \geq w(j)$, so for all $w \in G$, either $0 \leq y(j) \leq w(j)$ or $0 \geq y(j) \geq w(j)$. Theorem 2 implies that if $y \in G$ then y is uniquely rescale-optimal.

Conversely, suppose that there exists a uniquely rescale-optimal element u in G; we will prove that $y \in G$. Consider arbitrary $j \in \{1, \ldots, n\}$. The fact that G is a polyhedron implies that there exists some $w \in G$ with $w(j) = y(j)$. If $y(j) > 0$ then we know by Lemma 5 that $u(j) \geq y(j)$. But Theorem 2 implies that $u(j) \leq w(j) = y(j)$, and thus $y(j) = u(j)$. Similarly, if $y(j) < 0$ then $y(j) \geq u(j) \geq w(j) = y(j)$ and so, $y(j) = u(j)$. If $y(j) = 0$ then $w(j) = 0$, so $u(j) = 0$, also using Theorem 2. We have shown that $y = u$, so $y \in G$. □

Corollary 1 leads immediately to an algorithm for determining if $G(\Lambda, \Theta)$ has a uniquely rescale-optimal element, and finding it, if it exists. The algorithm involves at most $n+1$ runs of a linear programming solver, and thus determining and finding a uniquely rescale-optimal element u can be performed in polynomial time. If it succeeds in finding such a u then the induced preferences can be efficiently tested using: $\alpha \gg_{\Lambda,\Theta} \beta$ if and only if $u \cdot (\alpha - \beta) > 0$.

4 Zm-Pointwise Undominated Vectors

Proposition 2 states that being pointwise undominated is a necessary condition for being rescale-optimal. The example below shows that the two conditions are not equivalent. In this section we define a stronger version of pointwise undominated called *zm-pointwise undominated*, where 'zm' stands for *zeros-modified* (the essential difference being in the treatment of j such that $u(j) = 0$). We show that this is still a necessary condition for rescale-optimality, and is in fact equivalent to rescale-optimality (for polyhedra).

Example 2: Let $G \subseteq \mathbb{R}^2$ be given by all pairs (x, y) such that $x + y \geq 1$. It can be seen that the set of points that are pointwise undominated is $\{(x, 1 - x) : x \in [0, 1]\}$. On the other hand, the set of points that are rescale-optimal is $\{(x, 1 - x) : x \in (0, 1)\}$: neither $(1, 0)$ nor $(0, 1)$ is rescale-optimal. This is because if rescaling $\tau \in \mathbb{R}^2$ is such that $\tau(x)/\tau(y) = r$, for $r \in (0, \infty)$, then the associated rescale-optimal $w_\tau(G)$ is equal to $\frac{1}{1+r^2}(1, r^2)$, which is never equal to $(1, 0)$ or $(0, 1)$.

Definition 4 (zm-pointwise undominated) *We say that u is zm-pointwise undominated in G if for all $v \in G$, either (a) $v(j) = u(j)$ for all $j \in \{1, \ldots, n\}$ such that $u(j) \neq 0$; or (b) there exists $k \in \{1, \ldots, n\}$ such that either $0 < u(k) < v(k)$ or $0 > u(k) > v(k)$.*

It is easily shown using Lemma 2 that if u is zm-pointwise undominated in convex G then it is pointwise undominated in G. Proposition 3 below shows that being zm-pointwise undominated is a necessary condition for being rescale-optimal. The proof uses the following lemma.

Lemma 6 *Let $u, v \in \mathbb{R}^n$, with $u \neq v$, and let $\tau \in \mathbb{R}^n_+$. For $\delta \in (0, 1]$ let $v_\delta = \delta v + (1 - \delta)u$. Then the following hold:*

(i) For any $\delta \in \mathbb{R}$, $|v_\delta \odot \tau|^2 - |u \odot \tau|^2 = \delta^2 |(v - u) \odot \tau|^2 + 2\delta(\tau \odot \tau \odot u) \cdot (v - u)$.

(ii) $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$ *if and only if for all* $\delta \in (0,1]$, $|v_\delta \odot \tau| > |u \odot \tau|$.

(iii) There exists $\tau \in I\!R^n_+$ *such that* $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$ *if and only if either (a)* $v(j) = u(j)$ *for all* $j \in \{1,\ldots,n\}$ *such that* $u(j) \neq 0$*; or (b) there exists* $k \in \{1,\ldots,n\}$ *such that either* $0 < u(k) < v(k)$ *or* $0 > u(k) > v(k)$.

Proof: (i): Using $v_\delta = u + \delta(v-u)$, we have that $v_\delta \odot \tau = (u \odot \tau) + \delta(v-u) \odot \tau$. Then, $|v_\delta \odot \tau|^2 = (v_\delta \odot \tau) \cdot (v_\delta \odot \tau) = (u \odot \tau) \cdot (u \odot \tau) + \delta^2 |(v-u) \odot \tau|^2 + 2\delta (u \odot \tau) \cdot ((v-u) \odot \tau)$, which leads to the result.

(ii): If $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$ then (i) immediately implies that for all $\delta \in (0,1]$, $|v_\delta \odot \tau| > |u \odot \tau|$, since $|(v-u) \odot \tau|$ is non-zero (because $u \neq v$). Conversely, suppose that $(\tau \odot \tau \odot u) \cdot (v-u) < 0$. Choosing δ such that $\delta |(v-u) \odot \tau|^2 \leq -2(\tau \odot \tau \odot u) \cdot (v-u)$ gives, using (i), that $|v_\delta \odot \tau| \leq |u \odot \tau|$, proving (ii).

(iii), $\Rightarrow$: Suppose that there exists $\tau \in I\!R^n_+$ such that $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$, and it is not the case that there exists $k \in \{1,\ldots,n\}$ such that either $0 < u(k) < v(k)$ or $0 > u(k) > v(k)$. Then, we can see that for each $j \in \{1,\ldots,n\}$, $(\tau \odot \tau \odot u)(j)(v-u)(j) = \tau(j)^2 u(j)(v(j)-u(j)) \leq 0$ (since it clearly holds if $u(j) = 0$; if $u(j) > 0$ then $v(j) \leq u(j)$ so it also holds; if $u(j) < 0$ then $v(j) \geq u(j)$ so it holds then too). The sum (over each j) of these n terms is at least zero, since it is equal to $(\tau \odot \tau \odot u) \cdot (v-u)$ and thus, for each $j \in \{1,\ldots,n\}$, $(\tau \odot \tau \odot u)(j)(v-u)(j) = 0$. This implies that $v(j) = u(j)$ for all $j \in \{1,\ldots,n\}$ such that $u(j) \neq 0$.

$\Leftarrow$: If (a) $v(j) = u(j)$ for all $j \in \{1,\ldots,n\}$ such that $u(j) \neq 0$ then $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$. Now, assume that (b) holds, i.e., there exists $k \in \{1,\ldots,n\}$ such that either $0 < u(k) < v(k)$ or $0 > u(k) > v(k)$. For $\epsilon > 0$, define τ_ϵ by $\tau_\epsilon(k) = \sqrt{1+\epsilon}$, and $\tau_\epsilon(j) = \sqrt{\epsilon}$ for all $j \neq k$. Then, $\tau \odot \tau \odot u = u(k) e_k + \epsilon u$, so $(\tau \odot \tau \odot u) \cdot (v-u) = u(k)(v(k)-u(k)) + \epsilon u \cdot (v-u)$, which is greater than zero for sufficiently small ϵ, since $u(k)(v(k)-u(k)) > 0$. □

Proposition 3 *Let* u *be an element of convex* $G \subseteq I\!R^n$. *Then:*

(i) u *is rescale-optimal in* G *if and only if there exists* $\tau \in I\!R^n_+$ *such that for all* $v \in G$, $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$.

(ii) u *is zm-pointwise undominated in* G *if and only if for all* $v \in G$, *there exists* $\tau \in I\!R^n_+$ *such that* $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$.

(iii) If u *is rescale-optimal in* G *then* u *is zm-pointwise undominated in* G.

Proof: (i): Using Lemma 3, u is rescale-optimal in G if and only if there exists $\tau \in I\!R^n_+$ such that for all $v \in G - \{u\}$, $|v \odot \tau| > |u \odot \tau|$, which, since G is convex, is if and only if, there exists $\tau \in I\!R^n_+$ such that for all $v \in G - \{u\}$ and for all $\delta \in (0,1]$, $|v_\delta \odot \tau| > |u \odot \tau|$, where $v_\delta = \delta v + (1-\delta)u$. By Lemma 6(ii), this is if and only if there exists $\tau \in I\!R^n_+$ such that for all $v \in G - \{u\}$, $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$, which holds iff for all $v \in G$, $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$.

(ii) By Lemma 6(iii), u is zm-pointwise undominated in G if and only if for all $v \in G - \{u\}$, there exists $\tau \in I\!R^n_+$ such that $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$, from which (ii) follows.

(iii) follows immediately from (i) and (ii). □

Definition 5 (agreeing on signs) *For* $u, v \in I\!R^n$, *we say that* u and v agree on signs *if for all* $j = 1,\ldots,n$, *(i)* $u(j) = 0 \iff v(j) = 0$*; (ii)* $u(j) > 0 \iff v(j) > 0$*; and thus also: (iii)* $u(j) < 0 \iff v(j) < 0$.

Lemma 7 *If* u *and* v *agree on signs and* $u \neq \mathbf{0}$ *then* $u \cdot v > 0$.

Proof: Because $u \neq \mathbf{0}$, there exists $k \in \{1,\ldots,n\}$ with $u(k) \neq 0$. Then, since u and v agree on signs, $v(k)$ is non-zero and the same sign as $u(k)$, so $u(k)v(k) > 0$. Also, for any $j \in \{1,\ldots,n\}$, $u(j)v(j) \geq 0$, and thus, $u \cdot v > 0$. □

Proposition 3(i) implies the following characterisation of rescale-optimality, by letting $\mu' = \tau \odot \tau \odot u$ and $\mu = \frac{\mu'}{\mu' \cdot u}$; for the converse, we use τ such that $\mu = \tau \odot \tau \odot u$, so that $\tau(j)^2 = \mu(j)/u(j)$ when $u(j) \neq 0$.

Theorem 3 *Consider any* u *in convex* $G \subseteq I\!R^n$. *If* $u = \mathbf{0}$ *then it is the unique rescale-optimal element of* G. *Otherwise,* u *is rescale-optimal in* G *if and only if there exists* $\mu \in I\!R^n$ *agreeing on signs with* u *such that* $\mu \cdot u = 1$ *and for all* $w \in G$, $\mu \cdot w \geq 1$.

It turns out that being zm-pointwise undominated is equivalent to being rescale-optimal, for a polyhedron, see Theorem 4 below. We first give some lemmas that we will use in the proof.

We can write any polyhedron as $G_I = \{w \in I\!R^n : \forall i \in I, w \cdot \lambda_i \geq a_i\}$, for finite I, and with each $\lambda_i \in I\!R^n$ and $a_i \in I\!R$. We also consider $G_{J_u} = \{w \in I\!R^n : \forall i \in J_u, w \cdot \lambda_i \geq a_i\}$, where $J_u = \{i \in I : \lambda_i \cdot u = a_i\}$.

Lemma 8 *Consider any* $v \in G_{J_u}$. *Then there exists* $\delta' \in (0,1)$ *such that for all* δ *with* $0 < \delta \leq \delta'$, $\delta v + (1-\delta)u \in G_I$.

Proof: Let $x = v - u$, and, for all $\delta \in (0,1)$, let $v_\delta = u + \delta x = \delta v + (1-\delta)u$. Since $u, v \in G_{J_u}$, we have $v_\delta \in G_{J_u}$ for all $\delta \in (0,1)$. We will next show that for all sufficiently small δ, $v_\delta \in G_I$, i.e., that for all $i \in I$, $v_\delta \cdot \lambda_i \geq a_i$. Since, $v_\delta \in G_{J_u}$, this holds for all $i \in J_u$. Now, consider any $i \in I - J_u$. By definition of J_u we have $u \cdot \lambda_i > a_i$. This implies that there exists $\delta_i > 0$ with for all δ with $0 < \delta \leq \delta_i$, $(u \cdot \lambda_i) + \delta(x \cdot \lambda_i) > a_i$, and thus $v_\delta \cdot \lambda_i > a_i$. Let us choose $\delta' = \min\{\delta_i : i \in I - J_u\}$. Then for all δ with $0 < \delta \leq \delta'$, and for all $i \in I - J_u$, $v_\delta \cdot \lambda_i > a_i$, so for all $i \in I$, $v_\delta \cdot \lambda_i \geq a_i$, which shows that $v_\delta \in G_I$.

Lemma 9 *Consider non-zero* $u \in G_I$ *(as defined above). Then* u *is zm-pointwise undominated in* G_I *if and only if* u *is zm-pointwise undominated in* G_{J_u}.

Proof: $\Rightarrow$: Suppose that u is zm-pointwise undominated in G_I. Consider any $v \in G_{J_u}$. By Lemma 8, there exists $\delta \in (0,1)$ such that $v_\delta \in G_I$, where $v_\delta = \delta v + (1-\delta)u = u + \delta(v-u)$. Proposition 3(ii) implies that there exists $\tau \in I\!R^n_+$ such that $(\tau \odot \tau \odot u) \cdot (v_\delta - u) \geq 0$, i.e., $(\tau \odot \tau \odot u) \cdot (\delta(v-u)) \geq 0$, which implies that $(\tau \odot \tau \odot u) \cdot (v-u) \geq 0$. Applying Proposition 3(ii) again gives that u is zm-pointwise undominated in G_{J_u}.

$\Leftarrow$: This is immediate because $G_I \subseteq G_{J_u}$. □

Lemma 10 *$G_{J_u} + \{-u\}$ is equal to $\{\lambda_i : i \in J_u\}^*$.*

Proof: $v \in G_{J_u} + \{-u\}$ if and only if for all $i \in J_u$, $(v + u) \cdot \lambda_i \geq a_i$, which is if and only if for all $i \in J_u$, $v \cdot \lambda_i \geq 0$ (since, by definition, $u \cdot \lambda_i = a_i$ for all $i \in J_u$), which is if and only if $v \in (\{\lambda_i : i \in J_u\})^*$. □

Theorem 4 *Let u be an element of polyhedron $G \subseteq \mathbb{R}^n$. Then, u is rescale-optimal in G if and only if u is zm-pointwise undominated in G.*

Proof: G is a polyhedron, so, by definition, it can be written as $\{w \in \mathbb{R}^n : \forall i \in I, w \cdot \lambda_i \geq a_i\}$. Let $J_u = \{i \in I : \lambda_i \cdot u = a_i\}$, and let $G_{J_u} = \{w \in \mathbb{R}^n : \forall i \in J_u, w \cdot \lambda_i \geq a_i\}$. Proposition 3(iii) implies that if u is rescale-optimal in G then u is zm-pointwise undominated. We next prove the converse.

Assume that u is zm-pointwise undominated in G. Let $C = G_{J_u} + \{-u\}$. It follows easily, by Lemma 10, that $C = \{\lambda_i : i \in J_u\}^*$, which is a polyhedral cone, and thus, by the Minkowski-Weyl theorem (see e.g., Theorem 4.18 of [Gallier, 2008]), is a finitely generated convex cone, so we can write $C = co(W)$ for some finite set $W = \{w_1, \ldots, w_l\}$.

Let $C' = co(S)$ be the convex cone generated by $S = W \cup S_Z$ where $S_Z = \{e_j, -e_j : j \in Z\}$, and $e_j \in \mathbb{R}^n$ is the unit vector in the jth dimension, and $Z = \{j \in \{1, \ldots, n\} : u(j) = 0\}$. Also, let $T = E_+ \cup E_- \cup R$, where $E_+ = \{-e_j : u(j) > 0\}$, and $E_- = \{e_j : u(j) < 0\}$, and $R = \{-w_i : i \in M\}$, and where $M = \{i \in L : -w_i \notin C'\}$ and $L = \{1, \ldots, l\}$. Let H be the convex hull of T.

We will show that the assumption that u is zm-pointwise undominated implies that C' and H are disjoint. If there exists $h \in C' \cap H$ then h can be written as $w + v_0$ where $w \in C$ and $v_0 \in co(S_Z)$. Also, since $h \in H$, it can be written as $v_+ + v_- + y$, where $v_+ \in co(E_+)$, $v_- \in co(E_-)$ and $y \in co(R)$. (More specifically, for some $q_1, q_2, q_3 \in [0, 1]$ with $q_1 + q_2 + q_3 = 1$ we have $v_+ = q_1 v'_+$ for some v'_+ in the convex hull of E_+, and $v_- = q_2 v'_-$ for some v'_- in the convex hull of E_-, and $y = q_3 z$ for some z in the convex hull of R.) Since $-y \in C$, $w - y \in C$. Let $v = w - y + u = -v_0 + v_+ + v_- + u$. Then $v \in G_{J_u}$, because $v - u = w - y \in C$. For $j \in \{1, \ldots, n\}$, if $u(j) > 0$ then $v_0(j) = v_-(j) = 0$, so $v(j) = u(j) + v_+(j) \leq u(j)$. Similarly, if $u(j) < 0$ then $v(j) = u(j) + v_-(j) \geq u(j)$. Since, u is zm-pointwise undominated in G, u is zm-pointwise undominated in G_{J_u}, by Lemma 9. This implies that for all $j \in \{1, \ldots, n\}$, if $u(j) \neq 0$ then $v(j) = u(j)$, and thus $v_+(j) = v_-(j) = 0$, and so, $v_+ = v_- = \mathbf{0}$. This implies that $w + v_0 = y$ and $y \in H$. Also, since $\mathbf{0}$ is neither in the convex hull of E_+ nor E_-, we have $q_1 = q_2 = 0$, and thus $q_3 = 1$, and so, y is in the convex hull of R. By definition of convex hull, we can write y as $\sum_{i \in M} t_i(-w_i)$, with each $t_i \geq 0$, and for some $k \in M$, $t_k > 0$. Then $-t_k w_k = w + \sum_{i \in M, i \neq k} t_i w_i + v_0$. The right-hand-side is in $co(S)$, which implies that $-w_k \in C'$, which contradicts $k \in M$. Thus, C' and H are disjoint.

Both C' and H are convex and closed, and H is compact. A strict separating hyperplane theorem (see e.g., Theorem 2.1.5 of [Bot *et al.*, 2009]) implies that there exists vector $\mu \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that for all $g \in C'$, $\mu \cdot g > c$ and for all $h \in H$, $\mu \cdot h < c$. Since $\mathbf{0} \in C'$, we have $\mu \cdot \mathbf{0} > c$, so $c < 0$.

It can be shown that if g and $-g$ are both in C' then $\mu \cdot g = 0$. (Else $\mu \cdot g < 0$ or $\mu \cdot (-g) < 0$; without loss of generality assume $\mu \cdot g < 0$; then there exists $r > 0$ such that $\mu \cdot (rg) = r(\mu \cdot g) < c$, which contradicts $rg \in C'$.) This implies that if $u(j) = 0$ (so $j \in Z$ and $e_j, -e_j \in C'$) then $\mu \cdot e_j = 0$ and thus $\mu(j) = 0$. Also, if $i \in L - M$, then $w_i, -w_i \in C'$, so $\mu \cdot w_i = 0$. For any $i \in M$, we have that $-w_i \in H$, so $\mu \cdot (-w_i) < c < 0$, so $\mu \cdot w_i > 0$. Thus for any $w_i \in W$, $\mu \cdot w_i \geq 0$, and therefore for any $w \in C$, $\mu \cdot w \geq 0$, since w is a positive linear combination of the elements of W.

If $u(j) > 0$, then $-e_j \in H$, so $\mu \cdot e_j > -c > 0$, so $\mu(j) > 0$. Similarly, if $u(j) < 0$ then $\mu(j) < 0$. Thus, μ agrees on signs with u. This implies that $\mu \cdot u > 0$ (since $u \neq \mathbf{0}$), we let $\mu' = \frac{\mu}{\mu \cdot u}$, and thus $\mu' \cdot u = 1$.

For any $v \in G$, $v \in G_{J_u}$, and so $v - u$ is in C; we have shown that $\mu' \cdot (v - u) \geq 0$, so $\mu' \cdot v \geq \mu' \cdot u$. Theorem 3 then implies that u is rescale-optimal in G. □

5 Computational Characterisation of Rescale-Optimal

Here we extend the characterisation of rescale-optimality given in Theorem 4, leading to a computational method for testing rescale-optimality, and thus to a method for testing if $\alpha \gg_{\Lambda,\Theta} \beta$, for $\alpha, \beta \in \mathbb{R}^n$.

5.1 Expressing Rescale-Optimality in Terms of Positive Linear Combinations

Theorem 3 implies that non-zero u is rescale-optimal in $G(\Lambda, \Theta)$ if and only if there exists a vector μ that agrees on signs with u with $\mu \cdot w \geq \mu \cdot u$ for all $w \in G$. The main result of this section is the following, that shows that μ is a positive linear combination of certain vectors in $\Lambda \cup \Theta$.

Theorem 5 *Let G be a polyhedron, which we write as $G_I = \{w \in \mathbb{R}^n : \forall i \in I, w \cdot \lambda_i \geq a_i\}$, for finite I, and with each $\lambda_i \in \mathbb{R}^n$ and $a_i \in \mathbb{R}$. Consider any non-zero vector u in G_I. Then, u is rescale-optimal in G_I if and only if there exists $\mu \in \mathbb{R}^n$ that agrees on signs with u such that $\mu \cdot u = 1$ and μ is a positive linear combination of elements of $\{\lambda_i : i \in J_u\}$, where $J_u = \{i \in I : \lambda_i \cdot u = a_i\}$.*

Note that this implies that if non-zero u is rescale-optimal in G_I then J_u is non-empty, since $\mathbf{0}$ is the only positive linear combination of the empty set, and $\mu \neq \mathbf{0}$.

The proof uses the following lemmas. We first give a property that follows easily from standard results about convex cones.

Lemma 11 *Let Λ be a finite subset of $\mathbb{R}^n$ and let $\mu \in \mathbb{R}^n$. Then $\Lambda^* \subseteq (\{\mu\})^*$ if and only if $\mu \in co(\Lambda)$.*

Proof: Because $\Lambda^* = (co(\Lambda))^*$ we have that $\Lambda^* \subseteq (\{\mu\})^*$ if and only if $(co(\Lambda))^* \subseteq (co(\{\mu\}))^*$. Now, clearly, if $\mu \in co(\Lambda)$ then $(co(\Lambda))^* \subseteq (\{\mu\})^*$, and thus $\Lambda^* \subseteq (\{\mu\})^*$. To prove the converse, it is sufficient to show that $(co(\Lambda))^* \subseteq$

$(co(\{\mu\}))^*$ implies $\mu \in co(\Lambda)$. Now, $(co(\Lambda))^* \subseteq (co(\{\mu\}))^*$ implies $(co(\Lambda))^{**} \supseteq (co(\{\mu\}))^{**}$. Convex cones $co(\Lambda)$ and $co(\{\mu\})$ are both topologically closed, so, by a fundamental result for convex cones, $(co(\Lambda))^{**} = co(\Lambda)$ and $(co(\{\mu\}))^{**} = co(\{\mu\})$, and thus $co(\{\mu\}) \subseteq co(\Lambda)$, which implies that $\mu \in co(\Lambda)$. □

Lemma 12 *Consider non-zero $u \in G_I$ (as defined above). Then u is rescale-optimal in G_I if and only if u is rescale-optimal in $G_{J_u} = \{w \in I\!R^n : \forall i \in J_u, w \cdot \lambda_i \geq a_i\}$.*

Proof: This follows immediately from Theorem 4 and Lemma 9, since G_I and G_{J_u} are polyhedra. □

Proof of Theorem 5

First consider $\mu \in I\!R^n$ such that $\mu \cdot u = 1$. Then it can be seen that $\{w : w \cdot \mu \geq 1\} + \{-u\} = (\{\mu\})^*$. Also, $G_{J_u} \subseteq \{w : w \cdot \mu \geq 1\}$ if and only if $G_{J_u} + \{-u\} \subseteq (\{\mu\})^*$ $\iff$ $\{\lambda_i : i \in J_u\}^* \subseteq (\{\mu\})^*$, using Lemma 10, which, by Lemma 11, is if and only if, $\mu \in co(\{\lambda_i : i \in J_u\})$.

By Lemma 12, u is rescale-optimal in G_I if and only if u is rescale-optimal in G_{J_u}, which, by Theorem 3, is if and only if there exists $\mu \in I\!R^n$ agreeing on signs with u such that $\mu \cdot u = 1$ and $G_{J_u} \subseteq \{w : w \cdot \mu \geq 1\}$, i.e., $\mu \in co(\{\lambda_i : i \in J_u\})$, by the earlier argument. □

We have the following corollary (using the same notation), which shows that testing if u is rescale-optimal in G_I can be performed in polynomial time: by first checking that $u \in G_I$ (i.e., for all $i \in I$, $u \cdot \lambda_i \geq a_i$), and then testing if a set of inequalities has a solution, using a linear programming solver.

Corollary 2 *u is rescale-optimal in G_I if and only if $u \in G_I$ and there exists non-negative reals r_i for each $i \in J_u$, (i.e., $i \in I$ such that $\lambda_i \cdot u = a_i$) and vector $\tau \in I\!R^n$ with for all $j \in \{1, \ldots, n\}$, $\tau(j) \geq 1$, and $\tau(j)u(j) = \sum_{i \in J_u} r_i \lambda_i$.*

Proof: First suppose that u is rescale-optimal in G_I. Then $u \in G_I$, and, by Theorem 5, there exists $\mu \in I\!R^n$ that agrees on signs with u such that $\mu \cdot u = 1$ and there exist non-negative $r'_i \in I\!R$ such that $\mu = \sum_{i \in J_u} r'_i \lambda_i$. For all $j \in \{1, \ldots, n\}$ such that $u(j) \neq 0$, define $t_j = \mu(j)/u(j)$, which is greater than zero, because μ and u agree on signs, and let t be the minimum of these values. Define τ by $\tau(j) = 1$ if $u(j) = 0$ and otherwise, define $\tau(j) = t_j/t$. Then for all $j \in \{1, \ldots, n\}$, $\tau(j) \geq 1$, and $\tau(j)u(j) = \mu(j)/t = \sum_{i \in J_u} (r'_i/t)\lambda_i$.

Conversely, suppose that $u \in G_I$ and there exists non-negative reals r_i for each $i \in J_u$ and vector $\tau \in I\!R^n$ with for all $j \in \{1, \ldots, n\}$, $\tau(j) \geq 1$, and $\tau(j)u(j) = \sum_{i \in J_u} r_i \lambda_i$. Define $\mu \in I\!R^n$ to be $\frac{\tau \odot u}{(\tau \odot u) \cdot u}$. Then $\mu \cdot u = 1$, and μ agrees on signs with u, and is a positive linear combination of $\{\lambda_i : i \in J_u\}$. □

5.2 Computation of Inference

For finite subsets Λ, Θ of $I\!R^n$, and arbitrary $\alpha, \beta \in I\!R^n$, we would like to be able to determine if $\alpha \gg_{\Lambda,\Theta} \beta$. Now, $\alpha \not\gg_{\Lambda,\Theta} \beta$ if and only if there exists u that is rescale-optimal in $G(\Lambda, \Theta)$ such that $u \cdot (\beta - \alpha) \geq 0$. Labelling Λ as $\{\lambda_i : i \in I\}$ and Θ as $\{\theta_k : k \in K\}$, it follows, using Theorem 5, that $\alpha \not\gg_{\Lambda,\Theta} \beta$ if and only if there exists $u \in I\!R^n$ and $\mu \in I\!R^n$, non-negative reals r_i for each $i \in I$ and non-negative reals s_k for $k \in K$, such that

- $u \cdot (\beta - \alpha) \geq 0$;
- $\forall i \in I$, $u \cdot \lambda_i \geq 1$, and $[u \cdot \lambda_i = 1$ or $r_i = 0]$;
- $\forall k \in K$, $u \cdot \theta_k \geq 0$, and $[u \cdot \theta_k = 0$ or $s_k = 0]$;
- $\forall j = 1, \ldots, n$, $u(j) = 0 \iff \mu(j) = 0$, and $u(j) > 0 \iff \mu(j) > 0$; and
- $\mu = \sum_{i \in I} r_i \lambda_i + \sum_{k \in K} s_k \theta_k$.

6 Experimental Testing

The experiments make use of a subset of a year's worth of real ridesharing records, provided by a commercial ridesharing system *Carma* (see `http://carmacarpool.com/`). We base our experiments on 13 benchmarks derived from this data-set. Each ridesharing alternative has 7 features, representing different aspects of a possible choice of match for a given user. Each benchmark corresponds to the inferred preferences of a different user. An input preference of alternative α_i over β_i leads to $\alpha_i - \beta_i$ being included in Λ. However, a pre-processing phase deletes some elements of Λ, in order to make it consistent (i.e., $\Lambda^> \neq \emptyset$), since in this paper we assume consistent preferences. (We assume no additional constraints, so $\Theta = \emptyset$.) More information about the data can be found in [Montazery and Wilson, 2016].

We randomly generate 100 pairs of (α, β), based on a uniform distribution for each feature. A pair (α, β) is called *decisive* for preference relation $\gg_\Lambda$ if either $\alpha \gg_\Lambda \beta$ or $\beta \gg_\Lambda \alpha$ hold, i.e., if α and β are comparable with respect to $\gg_\Lambda$; similarly, for $\gg^*_\Lambda$. (All 100 pairs turn out to be decisive for $>^{mm}_\Lambda$, as one would expect.) Table 1 shows the percentage of decisive pairs for $\gg_\Lambda$ and $\gg^*_\Lambda$, as well as the running time per request. CPLEX 12.6.2 is used as the solver on a computer facilitated by a Core i7 2.60 GHz processor and 8 GB RAM memory. Determining $\alpha \gg^*_\Lambda \beta$ is based on consistency of a set of linear constraints, so is fast. Testing $\alpha \gg_\Lambda \beta$ is performed using quite a simple CPLEX model based on the approach in Section 5.2. For example, the constraint $[u \cdot \lambda_i = 1$ or $r_i = 0]$ can be expressed in CPLEX as $(u \cdot \lambda_i == 1) + (r_i == 0) \geq 1$. (Each logical proposition is treated as an integer: 0 for false and 1 for true.) Our encoding is rather simple, and there may well be faster computation methods; however, for this problem the computation time was already adequate.

The results indicate that for these benchmarks, $\gg_\Lambda$ is much more decisive than $\gg^*_\Lambda$. At the same time, $\gg_\Lambda$ is not equal to the maximum margin relation $>^{mm}_\Lambda$, so, in each case, rescaling makes a difference. Testing preference with respect to $\gg_\Lambda$ can be performed in reasonable time, the slowest instance being Benchmark 13, with a mean query time of around 5.3 seconds, based on 134 input preferences in Λ.

7 Discussion

We have described a novel way of inducing a preference relation—by considering all possible rescalings when apply-

Table 1: A comparison, using 13 benchmarks, between preference relations $\gg_\Lambda$ and $\gg^*_\Lambda$.

	$\lvert\Lambda\rvert$	Decisive Pairs (%)		Time (msec)	
		$\gg_\Lambda$	$\gg^*_\Lambda$	$\gg_\Lambda$	$\gg^*_\Lambda$
1.	24	19	2	248	9
2.	29	94	0	1554	7
3.	31	18	0	421	6
4.	36	83	27	2478	6
5.	38	35	2	2123	5
6.	41	63	14	3006	5
7.	53	41	15	873	3
8.	55	97	11	1347	5
9.	62	54	1	1218	4
10.	94	62	6	2810	4
11.	127	67	9	3495	5
12.	129	77	0	1652	3
13.	134	68	19	5330	4
Avg	66	60	8	2217	5

ing a linear SVM-based approach for preference learning—and derived formal results that allow its computation. Our experimental results indicate that the relation can be computed in a reasonable time for significantly sized instances, and that the relation can be considerably different from both the maximum margin relation and a simple cone-based relation. The results are also very relevant for a more general situation where one had restrictions on a set of allowable rescalings. There are a number of directions for further work, including: extensions to the case of inconsistent input sets and to the computation of which alternatives among a set can be optimal with respect to some rescaling; analysis—along similar lines as in this paper—to the other natural forms of invariance of feature spaces. Finally, it would be interesting to consider the application and generalisation of our results to other convex optimisation problems.

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Appendix

Minimum norm result

The following is a very well-known result; the first part follows from fact that if two different vectors $v, w \in \mathbb{R}^n$ have equal norm then $\frac{1}{2}(v+w)$ has strictly smaller norm. Specifically, consider any $v, w \in \mathbb{R}^n$. Let $m = \frac{1}{2}(v+w)$, and let $d = \frac{1}{2}(v-w)$. Thus, $v = m + d$ and $w = m - d$. So, $|v|^2 = v \cdot v = m \cdot m + d \cdot d + 2m \cdot d$, and $|w|^2 = m \cdot m + d \cdot d - 2m \cdot d$. Hence, $|v|^2 + |w|^2 = 2|m|^2 + 2|d|^2$, and so, $|m|^2 = \frac{1}{2}(|v|^2 + |w|^2) - |d|^2$. Now suppose that v and w are both in convex G, and have the same norm, and are not equal. Then, $m \in G$ and d is not the zero vector so, $|d| > 0$, which implies that $|m| < |v| = |w|$. This proves that there can be at most one element in a convex set with minimal norm. For the second part, choose some $v \in G$ and consider $G' = \{w \in G : |w| \leq |v|\}$. This is compact, i.e., closed and bounded, and so its image R under the norm function (which is a continuous function) is compact and is thus a closed subset of the non-negative real numbers. Therefore R contains a least element r, and there exists some element $w \in G'$ with norm r, so w has minimum norm in G' and thus also in G.

Proposition 4 *There exists at most one element in convex G with minimum norm. There is exactly one if G is also topologically closed.*

A closed convex cone in $\mathbb{R}^n$ is a topologically closed subset of $\mathbb{R}^n$ that is closed under (pointwise) addition and non-negative scalar multiplication.

Theorem 6 *Let Λ be a finite subset of $\mathbb{R}^n$, and, let $C(\Lambda) = \{w \in \mathbb{R}^n : \forall \lambda \in \Lambda, w \cdot \lambda > 0\}$, and let $G(\Lambda) = \{w \in \mathbb{R}^n : \forall \lambda \in \Lambda, w \cdot \lambda \geq 1\}$. Define function $\mathrm{marg} : C(\Lambda) \to \mathbb{R}$ by $\mathrm{marg}(w) = \min_{\lambda \in \Lambda} \frac{w \cdot \lambda}{|w|}$. Let H be a topologically closed convex cone in $\mathbb{R}^n$. Suppose that $C(\Lambda) \cap H$ is non-empty. Then $G(\Lambda) \cap H$ is non-empty and there exists a unique element w^* in $G(\Lambda) \cap H$ with minimum norm, and for $w \in C(\Lambda) \cap H$, w maximises marg within $C(\Lambda) \cap H$ if and only if w is a strictly positive scalar multiple of w^*, i.e., there exists $r \in \mathbb{R}$ with $r > 0$ such that $w = rw^*$.*

Lemma 13 *For $w \in C(\Lambda)$, define a_w to be $\min_{\lambda \in \Lambda} w \cdot \lambda$, (which is always non-zero by definition of $C(\Lambda)$), and define $\bar{w}$ to be $\frac{w}{a_w}$. Then, the following hold for any $w \in C(\Lambda)$.*

(i) $\bar{w} \in G(\Lambda)$.

(ii) If $w \in G(\Lambda)$ then $|w| > |\bar{w}|$ unless $w = \bar{w}$.

(iii) For any real $r > 0$, $\mathrm{marg}(rw) = \mathrm{marg}(w)$.

(iv) $\mathrm{marg}(w) = \frac{1}{|\bar{w}|}$.

Proof: Assume $w \in C(\Lambda)$. Then $a_{\bar{w}} = \min_{\lambda \in \Lambda} \frac{1}{a_w} w \cdot \lambda = \frac{a_w}{a_w} = 1$. Thus, $\bar{w} \in G(\Lambda)$, showing (i). Also, $\frac{|w|}{|\bar{w}|} = a_w$, by definition of $\bar{w}$. If $w \in G(\Lambda)$ then $a_w \geq 1$, so $|w| > |\bar{w}|$ unless $a_w = 1$, i.e., $w = \bar{w}$, proving (ii).

The definitions immediately imply that $\mathrm{marg}(w) = \frac{a_w}{|w|}$. Since $a_{\bar{w}} = 1$, we have $\mathrm{marg}(\bar{w}) = \frac{1}{|\bar{w}|}$. The definition of marg implies that for any real $r > 0$, $\mathrm{marg}(rw) = \mathrm{marg}(w)$, showing (iii), so, in particular, $\mathrm{marg}(w) = \mathrm{marg}(\bar{w}) = \frac{1}{|\bar{w}|}$, which proves (iv). □

Proof of Theorem 6

If $C(\Lambda) \cap H$ is non-empty then, by Lemma 13(i), $\bar{w} \in G(\Lambda)$ for any $w \in C(\Lambda)$. Also, $\bar{w} \in H$ since it is a positive scalar multiple of w, and H is a cone. Thus, $G(\Lambda) \cap H$ is non-empty. Proposition 4 implies that there exists a unique element w^* in $G(\Lambda) \cap H$ with minimum norm.

Consider any $w \in C(\Lambda) \cap H$. As we just showed, $\bar{w} \in G(\Lambda) \cap H$, so minimality of w^* implies that $|\bar{w}| \geq |w^*|$ which

equals $|\overline{w^*}|$, using Lemma 13(ii). Lemma 13(iv) then implies that $\text{marg}(w) \leq \text{marg}(w^*)$, which implies that w^* maximises marg in $C(\Lambda) \cap H$. Also, if $\text{marg}(w) = \text{marg}(w^*)$ then $|\bar{w}| = |w^*|$, and thus, $\bar{w} = w^*$ by uniqueness of w^*. Then $\frac{w}{a_w} = w^*$ so w is a positive scalar multiple of w^*. Finally, for any $r > 0$, $\text{marg}(rw^*) = \text{marg}(w^*)$ so rw^* maximises marg in $C(\Lambda) \cap H$. □

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